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From August 1983 to December 1986, I attended Florida State University for an AFIT sponsored PhD in statistics. Before I could complete my dissertation, I was assigned to AFIT/DFMS. On October 21, 1992 I successfully defended my dissertation and graduated on December 12, 1992. I have attached one copy of my dissertation and two copies of the abstract as you reqested in our telephone conversation of 28 July 1992.

Donna C Herge

DONNA C. HERGE, Lt Col, USAF
Director of Research and Analysis

EFFECTS OF INSPECTION ERROR ON OPTIMAL INSPECTION POLICIES AND SOFTWARE FAULT DETECTION MODELS

Donna Carol Herge, Ph.D.
Florida State University, 1992
Major Professors: Frank Proschan, Ph.D.
Jayaram Sethuraman, Ph.D.

Inspection policies are essential for many types of systems for which the status (functioning or failed) can be determined only by actual inspection. Two types of inspection error may occur. A functioning system may be incorrectly assessed as having failed or a failed system may be incorrectly assessed as functioning. These errors are designated as Types I and II respectively, and their impact on optimal inspection policies and software fault detection models is analyzed. For a periodic inspection model with Type I error, an optimal replacement age is obtained, then monotonicity and asymptotic properties of the long-run expected cost per unit of time are presented. Type I error is incorporated into a cumulative damage model. When the failure density is reverse rule of order 2, an algorithm to compute an optimal inspection sequence is derived, and it is proven that optimal intervals are increasing. Extending the optimal inspection sequence model of Barlow, Hunter, and Proschan to include Type II inspection error, it is proven that optimal intervals are nonincreasing for a PF_2 density,

and an algorithm to compute optimal intervals is derived. Additionally, monotonicity and majorization results are obtained for an optimal inspection sequence with Type II error. The impact of fault-detection error on a software optimal release time model is shown. The effect of fault diversity on the Jelinski-Moranda model and how this relates to imperfect fault detection is demonstrated.

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THE FLORIDA STATE UNIVERSITY

COLLEGE OF ARTS AND SCIENCES

EFFECTS OF INSPECTION ERROR ON OPTIMAL INSPECTION POLICIES AND SOFTWARE FAULT DETECTION MODELS

By

DONNA CAROL HERGE

A Dissertation submitted to the
Department of Statistics
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

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The members of the committee approve the dissertation of Donna Carol Herge defended on October 21, 1992.

Frank Proschan

Frank Proschan
Professor Codirecting Dissertation

J. Sethuraman

Jayaram Sethuraman
Professor Codirecting Dissertation

JR Quine

John R. Quine
Outside Committee Member

FW Leysieffer

Frederick W. Leysieffer
Committee Member

Thomas: Your arrival delayed this work's progress, but just as surely inspired its successful completion.

John: You are the wind beneath my wings.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Abstract

Inspection policies are essential for many types of systems for which the status (functioning or failed) can be determined only by actual inspection. Two types of inspection error may occur. A functioning system may be incorrectly assessed as having failed or a failed system may be incorrectly assessed as functioning. These errors are designated as Types I and II respectively, and their impact on optimal inspection policies and software fault detection models is analyzed. For a periodic inspection model with Type I error, an optimal replacement age is obtained, then monotonicity and asymptotic properties of the long-run expected cost per unit of time are presented. Type I error is incorporated into a cumulative damage model. When the failure density is reverse rule of order 2, an algorithm to compute an optimal inspection sequence is derived, and it is proven that optimal intervals are increasing. Extending the optimal inspection sequence model of Barlow, Hunter, and Proschan to include Type II inspection error, it is proven that optimal intervals are nonincreasing for a PF_2 density, and an algorithm to compute optimal intervals is derived. Additionally, monotonicity and majorization results are

obtained for an optimal inspection sequence with Type II error. The impact of fault-detection error on a software optimal release time model is shown. The effect of fault diversity on the Jelinski-Moranda model and how this relates to imperfect fault detection is demonstrated.

1. INTRODUCTION AND SUMMARY

1.1. Importance of Inspection Models.

Inspection policies are essential for many types of systems for which the status (functioning or failed) can be determined only by actual inspection. Safety and security systems are prime examples. Events such as the April 1986 nuclear reactor accident at Chernobyl and the December 1984 chemical leak at Bhopal remind us of the importance of safety devices in system design and of the need for optimal scheduling of inspection. Inspection policies also have applications in environmental control, medical screening, and inventory optimization.

An inspection error may be due to human error or malfunction of a detection device. A simple example is a fire extinguisher which is checked periodically. If the pressure gauge malfunctions, it may erroneously indicate the extinguisher is empty, and so the extinguisher is unnecessarily replaced.

The consequences of inspection error can be very costly. Failure to identify an O-ring problem resulted in the explosion of the space shuttle Challenger on January 28, 1986. Nine passengers were killed and 23 injured when

the failure of a cargo door latch ruptured the skin and a large section of the fuselage ripped away from the 19 year old United Airlines Boeing 747 enroute from Hawaii on February 25, 1989. Ironically the Federal Aviation Administration had ordered inspection of all 747 cargo door latches the previous year.

1.2 Imperfect Repair Models.

Recent work on maintenance models takes into account additional real-world factors such as "minimal" or "imperfect" repair action. Brown and Proschan (1983a) proposed several different imperfect maintenance and imperfect inspection models and (1983b) studied properties of the distribution of the time between perfect repairs. Fontenot and Proschan (1984) then developed optimal policies for several maintenance models based on the imperfect repair model of Brown and Proschan (1983b). Block, Borges, and Savits (1985) extended the results of Brown and Proschan (1983b) by allowing the probability of imperfect repair to depend on the age of the equipment. Chan and Sethuraman (1985) compared the expected number of failures by time t in several different minimal repair models. Nakagawa (1981) analyzed generalized models for determining the optimal number of minimal repairs before replacement.

1.3 Inspection Models.

Another real world factor, imperfect inspection, has not been considered as extensively in the literature. Many inspection models assume perfect inspection. Savage (1956) showed how to choose an optimal replacement procedure for a "preparedness" model, where equipment is kept in storage until an emergency occurs. Derman and Sacks (1960) solved the problem of choosing an optimal replacement rule for deteriorating equipment, where the amount of deterioration is observed periodically. Barlow, Hunter, and Proschan (1963) showed how to obtain optimum inspection schedules for a broad class of failure distributions. Taylor (1975), assuming a cumulative damage model for system failure, found an optimal replacement strategy that minimized the long-run expected cost per unit of time. Abdel-Hameed (1986) obtained an optimal periodic replacement policy for a system subject to shocks which are modeled by any counting process with jump size one. For systems which fail when cumulative continuous wear reaches a threshold, Park (1988) derived an optimal replacement policy under periodic inspections. Parmigiani (1990a) derived a Bayesian decision rule for optimal scheduling of inspections when the distribution of failure times is not fully known.

1.4 Type I and II Inspection Errors.

Although perfect inspection is often assumed, that is, no error is made in assessing the status of the system, two types of inspection error may actually occur. A functioning system may be incorrectly assessed as having failed or a failed system may be incorrectly assessed as functioning. Designate these errors as Type I and Type II respectively, analogous to Type I and Type II errors in hypothesis testing.

1.5. Imperfect Inspection Models.

Assuming Type II inspection error, Derman (1961) obtained the minimax inspection schedule for an unknown failure distribution. Weiss (1962) assumed Type II error in his periodic inspection model and obtained the asymptotic availability of a system with exponential failure distribution. Luss and Kander (1974) obtained optimal inspection policies for a system which continues to operate during inspection. They assumed that the failure of a system prior to an inspection is detected with probability one, but allowed for the occurrence of both Type I and Type II inspection errors during system checking. Kaio and Osaki (1984) introduced Type II error into Keller's (1974, 1982) models for near-optimum inspection schedules using an inspection density.

Assuming an exponential failure density, Sengupta

(1982) extended the model of Barlow, Hunter, and Proschan (1963) to obtain optimum inspection schedules with Type II inspection error. Parmigiani (1990d) further extended this model for any log-concave failure density with both Type I and Type II errors. A decision theoretic model for optimal screening ages for detection of chronic diseases was presented by Parmigiani (1990b,d). Herge, Proschan, and Sethuraman (1986) obtained an optimal replacement age under Type I inspection error. With Type II error, Parmigiani (1990c) obtained optimal one-test and two-test policies to avoid replacing a functioning unit.

1.6 Imperfect Detection of Software Faults.

Corresponding to the problem of inspecting hardware systems for failure, we have the problem of detecting faults in software systems. In today's world, the number and complexity of processes being controlled by computers is growing rapidly. As the size and complexity of computer programs increases, the probability that the software will function error-free is near zero. Thus it is essential to be able to measure the probability that software will perform its intended function at a specified time in a given environment, i.e., to measure software reliability. Failure of the software to perform its intended function may be due to many factors ranging from errors in the program code (bugs) to incorrect specification of program

requirements. Goel (1985) gives an overview of many of the analytical models that have been developed to measure software reliability, and provides a critical analysis of their underlying assumptions. Most models make use of the number of faults in the software as a basis to measure reliability.

The Jelinski-Moranda (JM) model (1972) is one of the first and simplest models and, therefore, probably most used. It assumes the failure rate of the software is proportional to the number of faults remaining in the software after a period of testing. Let N be the initial number of unknown software faults, and t_i be a time between discovery of the (i-1)st and ith fault, i=1,...,N. Then the failure rate is $r(t_i) = \phi[N-i+1]$, where $\phi > 0$ is a proportionality constant. The model further assumes that faults are independent and equally likely to cause software failures. It also assumes that detected faults are removed with certainty and no new faults are created. Also the occurrence of a fault is equivalent to the detection of the fault. This is analogous to the assumption of a perfect inspection process. These "perfect" assumptions are common to most software reliability models; however, they are rarely true in practice.

Goel and Okumoto (1978) present an imperfect debugging model which is an extension of the JM model, where p is the probability that a detected fault is corrected. Then the

failure rate at time t_i between the (i-1)st and ith failure is $r(t_i) = [N-p(i-1)]\phi$. The number of faults in the system is governed by a semi-Markov process with transition probabilities $Q_{ij}(t) = p[1-\exp(-i\phi t)]$. Bayesian prior estimates for N and ϕ are gamma distributed, and for p a beta prior is used; see Dunn and Ullman (1982).

The Shooman model, like the JM model assumes the failure rate is proportional to the number of remaining faults in the program. Specifically $r(t)=k[N/I - n(\tau)]$, where t is operating time of the system, τ is debugging time (independent of t), N is the initial number of faults, I is total number of instructions in the program, $n(\tau)$ is total number of faults corrected during τ (normalized by I), and k is a proportionality constant. This model also assumes perfect fault detection and correction. An extension of this model by Shooman and Natarajan (1977) allows for the introduction of new faults during the correction process.

One of the important applications of a software reliability model is in the decision process to determine when to end the testing phase of software development and to release the software for operational use. Okumoto and Goel (1980) present a procedure to minimize the cost of software in a release time model. In order to minimize the entire life-cycle cost of the software, based on the JM model, Bai and Yun (1988) determine the optimum number, n,

of errors that need to be corrected before releasing a software system to the operational phase. The more errors that are corrected during the testing phase, the higher the reliability and the lower the operating cost of the software. But a prolonged testing phase results in increased cost of the software. To achieve an optimum n, the average gain is used as a criterion. Bai and Yun conclude that the results of their model for the optimum number of errors to release software, are approximately the same as those obtained by the Koch and Kubat (1983) model which determines the optimum time, T, to release software based on the average gain function. Again as in most models, a basic assumption is made that faults are detected and corrected with certainty, taking negligible time.

1.7 Software Fault Diversity.

In addition to the assumption of perfect fault detection, many software models, including the JM model, assume that each fault is equally likely to cause a software failure. But in reality some faults are more likely to occur than others. Boland, Proschan, and Tong (1987) introduced a discrete software reliability model based on the multinomial distribution and demonstrated the impact of fault diversity on time to detection of faults. If the software initially contains N faults, then the

initial state of the system can be described by a vector, $\underline{\theta} = (\theta_1, \theta_2, \ldots \theta_N)$, where θ_i is the probability of detecting fault i. It is shown that the more diverse (in terms of majorization) $\underline{\theta}$ is, the longer it takes (stochastically) to eliminate faults. The impact of fault diversity is also demonstrated for models of the Jelinski-Moranda type, but not specifically for the Jelinski-Moranda model.

1.8 Summary.

In this dissertation we analyze the impact of imperfect inspection on several models. Chapters 2 and 3 present Type I inspection error models. Chapters 4, 5, and 6 present Type II inspection error models.

In Chapter 2 we obtain an optimal replacement age in an imperfect inspection model and give an example for a device with a truncated normal life distribution. We give monotonicity and asymptotic properties for the long-run expected cost per unit of time. In Chapter 3 we extend the cumulative damage model of Derman and Sacks (1960) for the case in which the probability of perfect inspection is less than one.

Chapter 4 extends the Barlow, Hunter, and Proschan (1963) optimal inspection sequence model to include reverse rule two (RR2) failure densities. Chapter 5 then extends the Barlow, Hunter, and Proschan (1963) model to include Type II error, and monotonicity and majorization results

are presented.

Chapter 6 presents two software fault detection models. Section 6.2 explores the impact of imperfect fault detection on the Koch and Kubat (1983) model. Section 6.3 shows the effect of fault diversity of the JM model and shows how this relates to imperfect fault detection.

2. MODEL 1: OPTIMAL REPLACEMENT AGE WITH TYPE I INSPECTION ERROR

2.1. Formulation of the Model; Derivation of Solution.

Herge, Proschan, and Sethuraman (1986) formulate and analyze a periodic inspection model with Type I inspection error for a system which is subject to an age replacement policy. We describe the model and present the results in this chapter. A list of basic symbols follows.

X = lifelength with distribution F, $\overline{F}(x) \equiv 1 - F(x)$.

k, 2k, ... are scheduled inspection times.

$f_k \equiv f(k)$, $\overline{F}_k \equiv \overline{F}(k)$, $r_k \equiv f_k/\overline{F}_k$.

T = replacement age (positive integer multiple of k).

c_1 = cost of unscheduled replacement.

c_2 = cost of scheduled (age) replacement ($c_1 > c_2$).

q_1 = probability of Type I inspection error.

$p_1 = 1 - q_1$.

Y = observed lifelength of device with lifelength X.

$L_{p_1}(T)$ = long-run expected cost per unit of time.

A device has lifelength X with distribution function F, which is absolutely continuous with density f and F(0)=0. The device is installed at time zero and is inspected at successive times k, 2k, ... to determine if it

is functioning or failed. We follow an age replacement policy which calls for the unit to be replaced by a new unit at age T (which is a positive integer multiple of k) or at the first inspection following failure, whichever comes first. Replacement time is assumed negligible. When an error in inspection (Type I) occurs (with probability $0 < q_1 \le 1$), a functioning unit is replaced by a new unit. Define $p_1=1-q_1$. On the other hand, a failed unit is replaced at inspection with probability one. This process is continued indefinitely. Clearly the process renews itself at times of replacement.

It is apparent that an age replacement policy is inappropriate if the underlying failure rate is decreasing. Thus we will assume that the failure rate $r(t)=f(t)/[\overline{F}(t)]$ is increasing. The probability that the device survives until time t=ik is $P[Y>ik]=\overline{F}_{ik}p_1^i$ for i=0,1,2,... in the absence of planned replacement, since the device must survive to time ik and pass i inspections, where Y is the observed lifelength of the device. Let Z be the elapsed time between replacements; then Z=min{Y,T}. Note that $E(Y) = k\sum_{i=0}^{\infty}\overline{F}_{ik}p_1^i$.

Let c_1 be the cost of an unscheduled replacement (resulting from actual or believed failure) and c_2 be the cost of a scheduled (age) replacement. Assume $c_1 > c_2$. Our objective is to find the replacement age T which

minimizes $L_{p_1}(T)$, the long-run expected cost per unit of time. Theorem 2.1 gives a formula for $L_{p_1}(T)$ and Theorem 2.2 describes the optimal value T^* of T.

Theorem 2.1.
$$L_{p_1}(T) = \frac{c_1 - (c_1 - c_2)\overline{F}_{T-k}\, p_1^{T/k-1}}{k\sum_{i=0}^{T/k-1} \overline{F}_{ik}\, p_1^i}.$$

Proof. The times of replacement are renewal times in a renewal-reward process whose interarrival distribution is that of Z = min {Y,T}. From renewal theory, we have that $L_{p_1}(T)$ = C(T)/D(T), where C(T) is the expected cost per renewal cycle, and D(T) is the expected duration of a renewal cycle. Denote the distribution function of Y by G. Then for i=1,2,...

$$\overline{G}(t) = P[Y>t] = \overline{F}_{(i-1)k}p_1^{i-1}, \quad (i-1)k \le t < ik.$$

Then D(T), the expected length of a cycle is

$$D(T) = E(Z) = \int_0^T \overline{G}(t)\,dt = \sum_{i=1}^{T/k} \int_{(i-1)k}^{ik} \overline{G}(t)\,dt$$
$$= \sum_{i=1}^{T/k} \overline{F}_{(i-1)k}\, p_1^{i-1} \int_{(i-1)k}^{ik} dt = k\sum_{i=0}^{\frac{T}{k}-1} \overline{F}_{ik}\, p_1^i.$$

With probability $\overline{F}_{T-k}\, p_1^{\frac{T}{k}-1}$, the device is replaced after time

T-k and a cost of c_2 is incurred. Otherwise, it is replaced at or before time T-k and a cost of c_1 is incurred. Therefore the expected cost per cycle is

$$C(T) = c_1(1 - \overline{F}_{T-k}\, p_1^{\frac{T}{k}-1}) + c_2\overline{F}_{T-k}\, p_1^{\frac{T}{k}-1}. \quad \|$$

Theorem 2.2. In addition to the assumptions about F given above, assume that r is differentiable. Then

a. $L'_{p_1}(T) = \overline{F}_{T-k}p_1^{T/k-1}[k\sum_{i=0}^{T/k-1}\overline{F}_{ik}p_1^i]^{-2}H(T)$, where

$$H(T) = (c_1-c_2)[(kr_{T-k}-\ln p_1)\sum_{i=0}^{T/k-1}\overline{F}_{ik}p_1^i + \overline{F}_{T-k}p_1^{T/k-1}] - c_1 \quad .$$

b. An optimal replacement age T^* exists (T^* may be infinite), and $T^*=k \Leftrightarrow p_1 \leq \exp(c_2/(c_2-c_1) + kr_0)$.

c. T^* is finite $\Leftrightarrow r_\infty > c_1/(c_1-c_2)EY + (1/k)\ln p_1$.

d. $EY/k < c_1/c_2$ implies that $T^* < \infty$.

e. A lower bound for T^* is $(c_2/c_1)EY$.

Proof.

a. Differentiate.

b. Given $c_1 > c_2$ note that H is increasing since r is increasing and $H'(T) = (c_1-c_2)kr'_{T-k}\sum_{i=0}^{\frac{T}{k}-1}\overline{F}_{ik}\, p_1^i$. Then $T^* = k \Leftrightarrow$ L_{p_1} is increasing $\Leftrightarrow H(k) \geq 0 \Leftrightarrow$

$(c_1-c_2)(kr_0-\ln p_1 + 1) - c_1 \geq 0 \Leftrightarrow p_1 \leq \exp(c_2/(c_2-c_1) + kr_0)$.

If $p_1 > \exp(c_2/(c_2-c_1) + kr_0)$, then $H(k) < 0$. Thus if we can find $T_0 = \min\{T: H(T) \geq 0\}$, then L'_{p_1} is negative on $(0, T_0-k)$ and positive on (T_0, ∞). So we choose T^* such that $L(T^*) = \min\{L(T_0-k), L(T_0)\}$. Otherwise, if there is no T such that $H(T) \geq 0$, then L is decreasing and $T^* = \infty$; i.e., planned replacement does not occur.

c. The inequality involving r_∞ holds if and only if $\lim_{T\to\infty} H(T) > 0$. Since $H(0) < 0$, the equation $T_0 = \min\{T: H(T) \geq 0\}$ has a finite solution.

d. Note that $L_{p_1}(T) \leq L_{p_1}(k) = c_2/k$. If $T^* = \infty$, then $L_{p_1}(T^*) = c_1/EY \leq c_2/k$. By the contrapositive, $EY/k < c_1/c_2$ implies that $T^* < \infty$.

e. It is easy to show that $L_{p_1}(T) \geq c_2/T$. Thus $L_{p_1}(T)$ must cross $L_{p_1}(\infty) = c_1/EY$ (if it crosses at all) to the right of the value at which c_2/T crosses c_1/EY; this value is $T = (c_2/c_1)EY$. Therefore, we should schedule replacement at an age greater than $(c_2/c_1)EY$. ‖

2.2. <u>Asymptotic and Monotonicity Properties of $Lp_1(T)$</u>.

If we let $p_1=1$, we get a desired asymptotic result for $L_1(T)$ as the length of the inspection interval, k, goes to zero. In this case the model reduces to the simple age

replacement model for which the optimization problem was solved by Barlow and Proschan (1965), pp. 85-90. If $\overline{F}$ is Riemann integrable, then since the denominator of $L_1(T)$ is a Riemann sum, it converges to the integral form given in Barlow and Proschan (1965) p. 88, as follows:

$$\lim_{k\to 0} L_1(T) = \lim_{k\to 0} \frac{c_2 + (c_1 - c_2) F_{T-k}}{\sum_{i=0}^{T/k-1} k\overline{F}_{ik}} = \frac{c_2 + (c_1 - c_2) F_T}{\int_0^T \overline{F}(x)\,dx}.$$

Next for $0<p_1<1$, we provide asymptotic results for $L_{p_1}(T)$ as the length of the inspection interval, k, goes to zero, and also as k goes to infinity.

<u>Theorem 2.3</u>. For $0 < p_1 < 1$, $\lim_{k\to 0} L_{p_1}(T) = \infty$, and $\lim_{k\to\infty} L_{p_1}(T) = 0$.

<u>Proof</u>. <u>Case 1</u>. For T = k, $L_{p_1}(k) = \frac{c_2}{k}$, then $\lim_{k\to 0} \frac{c_2}{k} = \infty$, and $\lim_{k\to\infty} \frac{c_2}{k} = 0$.

<u>Case 2</u>. For T = ∞, $L_{p_1}(\infty) = \frac{c_1}{EY} = \frac{c_1}{k\sum_{i=0}^{\infty} \overline{F}_{ik}\, p_1^i}$, then

$$\lim_{k\to 0} \frac{c_1}{k\sum_{i=0}^{\infty} \overline{F}_{ik}\, p_1^i} > \lim_{k\to 0} \frac{c_1}{k\sum_{i=0}^{\infty} p_1^i} = \lim_{k\to 0} \frac{c_1}{k\left(\dfrac{1}{1-p_1}\right)} = \infty,$$

and

$$\lim_{k\to\infty} \frac{c_1}{k\sum_{i=0}^{\infty} \overline{F}_{ik}\, p_1^i} = \lim_{k\to\infty} \frac{c_1}{k(1 + \sum_{i=1}^{\infty} \overline{F}_{ik}\, p_1^i)} = 0.$$

<u>Case 3</u>. For $k < T < \infty$, $T = mk$ for a positive integer m, and $L_{p_1}(mk) = \dfrac{c_1-(c_1-c_2)\overline{F}_{(m-1)k}\, p_1^{m-1}}{k\sum_{i=0}^{m-1} \overline{F}_{ik}\, p_1^i}$, then

$$\lim_{k\to 0} \frac{c_1-(c_1-c_2)\overline{F}_{(m-1)k}\, p_1^{m-1}}{k\sum_{i=0}^{m-1} \overline{F}_{ik}\, p_1^i} > \lim_{k\to 0} \frac{c_1-(c_1-c_2)\, p_1^{m-1}}{k\sum_{i=0}^{m-1} p_1^i} = \lim_{k\to 0} \frac{c_1-(c_1-c_2)\, p_1^{m-1}}{k\left(\dfrac{1-p_1^m}{1-p_1}\right)} = \infty,$$

and

$$\lim_{k\to\infty} \frac{c_1-(c_1-c_2)\overline{F}_{(m-1)k}\, p_1^{m-1}}{k\sum_{i=0}^{m-1} \overline{F}_{ik}\, p_1^i} = \lim_{k\to\infty} \frac{c_1}{k(1 + \sum_{i=1}^{m-1} \overline{F}_{ik}\, p_1^i)} = 0. \quad \|$$

At $p_1 = 1$, $\lim_{k\to 0} L_{p_1}(T)$ is discontinuous since

$$\lim_{k\to 0} L_1(T) = \frac{c_2 + (c_1-c_2)F_T}{\int_0^T \overline{F}(x)\,dx} < \infty$$

while for $0 < p_1 < 1$, $\lim_{k\to 0} L_{p_1}(T) = \infty$.

Theorem 2.3 shows that the more frequently you inspect, the greater the cost per unit of time. This logically results in the conclusion that it would be best not to inspect at all, but simply to perform a planned replacement at time T. However, there are situations in which a fixed k is reasonable and justifiable; e.g.: the government requires that a certain vital part of the airplane (like the recorder) is inspected every k hours, the fire alarm system in a home or factory is inspected every k hours, etc.

It is easily seen that $L_{p_1}(T)$ increases as $p_1 \downarrow 0$ as follows.

a. Note that $\overline{F}_{ik} p_{1m}^{\,i} \leq \overline{F}_{ik} p_{1n}^{\,i}$ for $p_{1m} < p_{1n}$, for i = 0,1,...,T/k - 1. Thus for T = k, 2k, ...

$$1/(k\sum_{i=0}^{\frac{T}{k}-1}\overline{F}_{ik}\,p_{1m}^{i}) \geq 1/(k\sum_{i=0}^{\frac{T}{k}-1}\overline{F}_{ik}p_{1n}^{i}).$$

When T > k, we have strict inequality above.

b. Note that for $p_{1m} < p_{1n}$ and T =k, 2k, ...

$$\overline{F}_{T-k}\,p_{1m}^{\frac{T}{k}-1} \leq \overline{F}_{T-k}\,p_{1n}^{\frac{T}{k}-1}.$$

When T > k we have strict inequality. Thus

$$c_1 - (c_1 - c_2)\overline{F}_{T-k}\,p_{1m}^{\frac{T}{k}-1} \geq c_1 - (c_1 - c_2)\overline{F}_{T-k}\,p_{1n}^{\frac{T}{k}-1}.$$

From a. and b. we have $L_{p_{1m}}(T) = L_{p_{1n}}(T)$ for T = k and $L_{p_{1m}}(T) > L_{p_{1n}}(T)$ for $p_{1m} < p_{1n}$ and T = 2k, 3k, ...
This is intuitively reasonable since more mistakes in inspection leads to an increase in unnecessary replacements. This can be seen in Figure 2.5 in the next section. Clearly when $p_1 \downarrow 0$ $\lim L_{p_1}(T) = c_1/k = L_0(T)$, thus $L_0(T)$ is an upper bound for $L_{p_1}(T)$ for $0 \le p_1 \le 1$.

Since $L_{p_1}(T)$ is a decreasing function of p_1 (see Figure 2.5), we obtain a sharp lower bound for $L_{p_1}(T)$ by taking the limit as p_1 increases to one. Thus

$$L_{p_1}(T) \ge \lim_{p_1 \uparrow 1} L_{p_1}(T) = \frac{c_1 - (c_1 - c_2)\overline{F}_{T-k}}{k \sum_{i=0}^{T/k-1} \overline{F}_{ik}} = L_1(T).$$

We denote the optimal value of T for a particular p_1 as $T^*_{p_1}$. Then the optimal long-run expected cost per unit of time is $L_{p_1}(T^*_{p_1}) = \min_T L_{p_1}(T)$. Since for all T and $p_{1m} < p_{1n}$ we have $L_{p_{1m}}(T) \ge L_{p_{1n}}(T)$, it follows that
$L_{p_{1m}}(T^*_{p_{1m}}) \ge L_{p_{1n}}(T^*_{p_{1m}}) \ge L_{p_{1n}}(T^*_{p_{1n}})$.

2.3. Application to the Exponential Distribution.

Our model makes sense only for a failure rate which is

increasing. We see from Theorem 2.1 that the denominator of $L_{p_1}(T)$ is a sum which can be explicitly solved for an exponential distribution F. Intuitively it is clear that the optimal policy is to replace only at failure times, because replacing an exponential component by a new component does not change the residual life. We show that Theorem 2.2 confirms that $T^* = \infty$. Let $\overline{F}(x) = \exp(-\lambda x)$, $\lambda > 0$. Then $EY = \sum_{i=0}^{\infty}\left(p_1 e^{-\lambda k}\right)^i = \dfrac{k}{1-p_1 e^{-\lambda k}}$. Setting $L'_{p_1}(T) = 0$ solving

for T we get

$$T = k + k\ \ln\left[\frac{\left(\dfrac{c_1}{c_1-c_2}\right)\left(1-p_1 e^{-\lambda k}\right) + \ln p_1 e^{-\lambda k}}{1+p_1 e^{-\lambda k}\left(-1 + \ln p_1 e^{-\lambda k}\right)}\right] \Big/ \ln p_1 e^{-\lambda k}.$$

Note that $1 + p_1 e^{-\lambda k}\ (-1 + \ln p_1 e^{-\lambda k}) > 0$ since it reduces to the form $\ln x < x - 1$, which is true for $x = p_1^{-1} e^{\lambda k} > 1$. Thus for T to exist, we need $\left(\dfrac{c_1}{c_1-c_2}\right)\left(1-p_1 e^{-\lambda k}\right) + \ln p_1 e^{-\lambda k} > 0$, which implies $r_\infty = \lambda < \dfrac{c_1}{(c_1-c_2)EY} + \dfrac{1}{k}\ln p_1$. Thus from Theorem 2.2.c. we see that $T^* = \infty$.

2.4. Application to the Truncated Normal Distribution.

An application of the model for a truncated normal distribution was chosen so that results can be compared to the asymptotic case treated in Example 1, p.90, of Barlow and Proschan (1965).

Example 2.1. A certain tube used in commercial airline communication equipment has a truncated normal failure distribution with a mean life of 9080 hours and a standard deviation of 3027. Suppose c_1= \$1100 and c_2= \$100.

The density f(x) of the truncated normal distribution may be written as

$$f(x) = \begin{cases} \frac{1}{b\sigma}\varphi(\frac{x-\mu}{\sigma}), & x \geq 0 \\ 0 & otherwise, \end{cases}$$

$$where\ \varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\ and\ b = \frac{1}{\sigma}\int_0^\infty \varphi(\frac{x-\mu}{\sigma})dx.$$

From L'(T) = 0 we get

$$(kr_{T-k}-\ln p_1)\sum_{i=0}^{\frac{T}{k}-1}\overline{F}_{ik}\,p_1^i + \overline{F}_{T-k}\,p_1^{\frac{T}{k}-1} = \frac{c_1}{c_1-c_2}. \qquad (2.1)$$

Making the change of variable u =(T-k-μ)/σ, we have from (2.1) for a truncated normal distribution F,

$$(\frac{k}{\sigma}r(u)-\ln p_1)\sum_{i=0}^{\frac{\sigma u+\mu}{k}} p_1^i\int_{\frac{ik-\mu}{\sigma}}^{\infty}\varphi(v)dv + p^{\frac{\sigma u+\mu}{k}}\int_u^\infty\varphi(v)dv = \frac{bc_1}{c_1-c_2}. \qquad (2.2)$$

Define K(u) to be the left hand side of (2.2).

If $\mu/\sigma \geq 3$, then f(x) is very close to the density of a normal distribution with mean μ and standard deviation σ. Thus, in Example 1, Barlow and Proschan estimate b by 1 and

graphically estimate u to be -1.5. To obtain more accurate results for comparison with our model, we compute b =.9987 and K(u) = 1.0986. Then u = -1.63, and the optimal replacement age is 4146 hours with an associated minimum cost of L(4146) = $.036.

Next we observe the impact of Type I inspection error on this example. We obtain T^* for four cases: when the length of the inspection interval is k=1000 and the probability of no error in inspecting a functioning unit is p_1=.95; when k=500 and p_1=.95; when k=1000 and p_1=.5; and when k=500 and p_1=.5. Note that for the truncated normal distribution $r_\infty = \infty$, so we know by Theorem 2.2.c. that T^* is finite.

Using (2.2) we will find the value of T such that $K((T-k-\mu)/\sigma)$ is nearest to $K(u) = bc_1/(c_1-c_2) = 1.0986$. Tables 1 through 4 show the results of these computations. The corresponding graphs of $L_{p_1}(T)$ are shown in Figures 1 through 4.

We see from Table 1 below that $T^* = 4000$ is the optimal value, which gives us the minimum cost $L_{.95}(T^*)$ = $.0710 as shown in Figure 1. We compute EY = 7629. Recall from the proof of Theorem 2.2.e. that $c_1/EY = .144$ is the horizontal asymptote for $L_{p_1}(T)$. Also $(c_2/c_1)EY = 694$ is a lower bound for T^*.

Table 1. Values of K(u) for k=1000, p_1=.95

T	u	K(u)
1000	-3.00	1.050
2000	-2.67	1.051
3000	-2.34	1.060
4000	-2.01	1.088
5000	-1.68	1.149

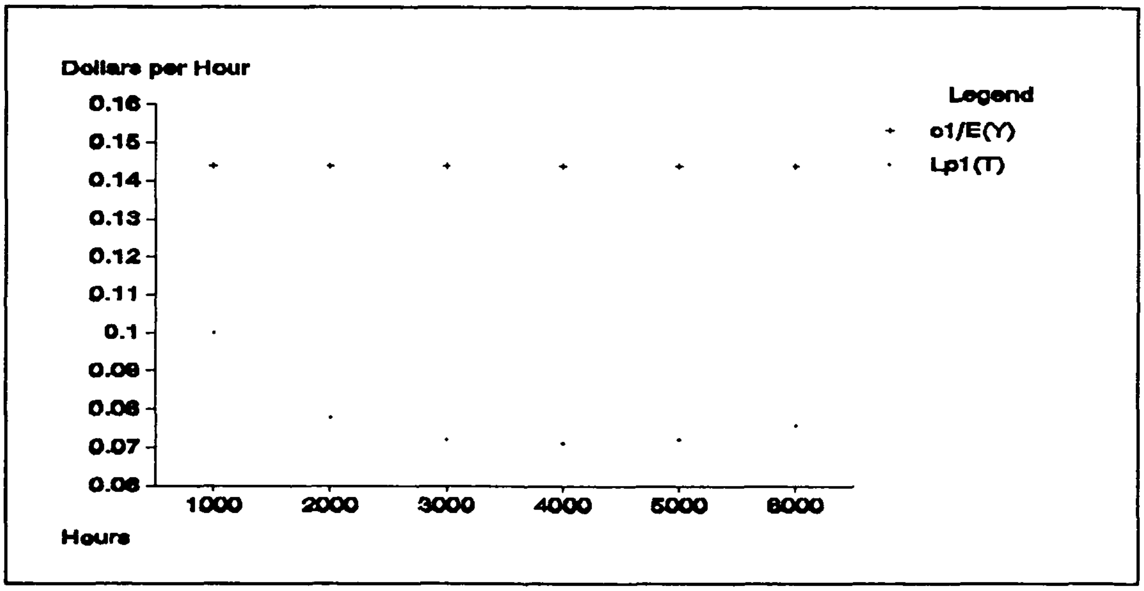

Figure 1. Values of $L_{p_1}(T)$ for k=1000, p_1=.95.

We see from Table 2 below that T^* = 4000 is the optimal value and $L_{p_1}(T^*)$ = \$.1271 shown in Figure 2. We compute EY = 5979. Then (c_2/c_1)EY = 544 is a lower bound for T^*. Notice in Figure 2 that $c_2/k > c_1/EY$; thus the implication in Theorem 2.2.d. cannot be reversed.

Comparing the cases where k=500 and k=1000, it is clear that more frequent inspections result in a higher optimal cost $L_{p_1}(T^*)$.

Table 2. Values of K(u) for k=500, p_1=.95

T	u	K(u)
3500	-2.01	1.0700
4000	-1.84	1.0916
4500	-1.65	1.1205
5000	-1.51	1.1605

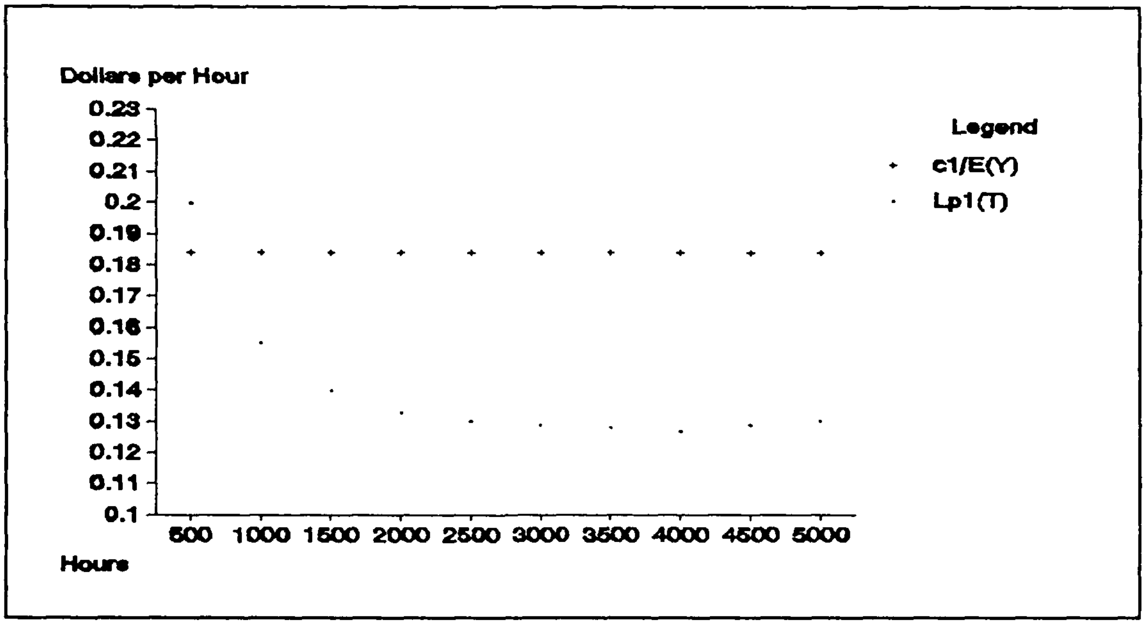

Figure 2. Values of $L_{p_1}(T)$ for k=500, p_1=.95.

We see from Table 3 below that the graph of K(u) does not cross $bc_1/(c_1-c_2)$ = 1.0986. Thus there is no solution

to $L'_{p_1}(T) = 0$, so that $L'_{p_1} > 0$, and the cost is strictly increasing, as shown in Figure 3.

Table 3. Values of K(u) for k=1000, p_1=.5

T	u	K(u)
1000	-3.00	1.6923
2000	-2.67	1.5410
3000	-2.34	1.4710
4000	-2.01	1.4490
5000	-1.68	1.4590
6000	-1.35	1.4969

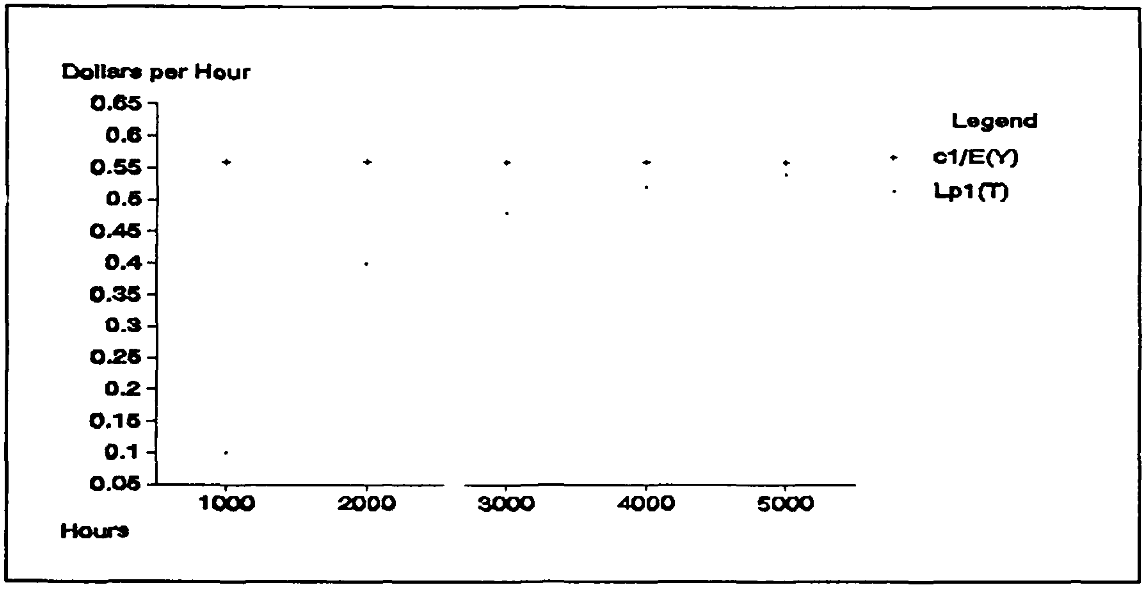

Figure 3. Values of $L_{p_1}(T)$ for k=1000, p_1=.5.

Then the minimum cost is \$.10 = $L_{p_1}(1000)$ at $T^* = k$. This is expected from Theorem 2.2.b. since $.5 = p_1 \leq$

$\exp(c_2/(c_2-c_1)+kr_0) = \exp(-.1 + 1000(.0000015)) \approx .9062$.
Since EY = 1980, (c_2/c_1)EY = 180 is a lower bound for T^*.

As in the previous case where p_1=.5, we see in Table 4 below that K(u) does not cross $bc_1/(c_1-c_2)$ = 1.0986, so that

Table 4. Values of K(u) for k=500, p_1=.5

T	u	K(u)	T	u	K(u)
500	-3.00	1.6915	3000	-2.17	1.4040
1000	-2.83	1.5384	3500	-2.01	1.4042
1500	-2.67	1.4630	4000	-1.84	1.4089
2000	-2.50	1.4260	4500	-1.68	1.4164
2500	-2.34	1.4096	5000		

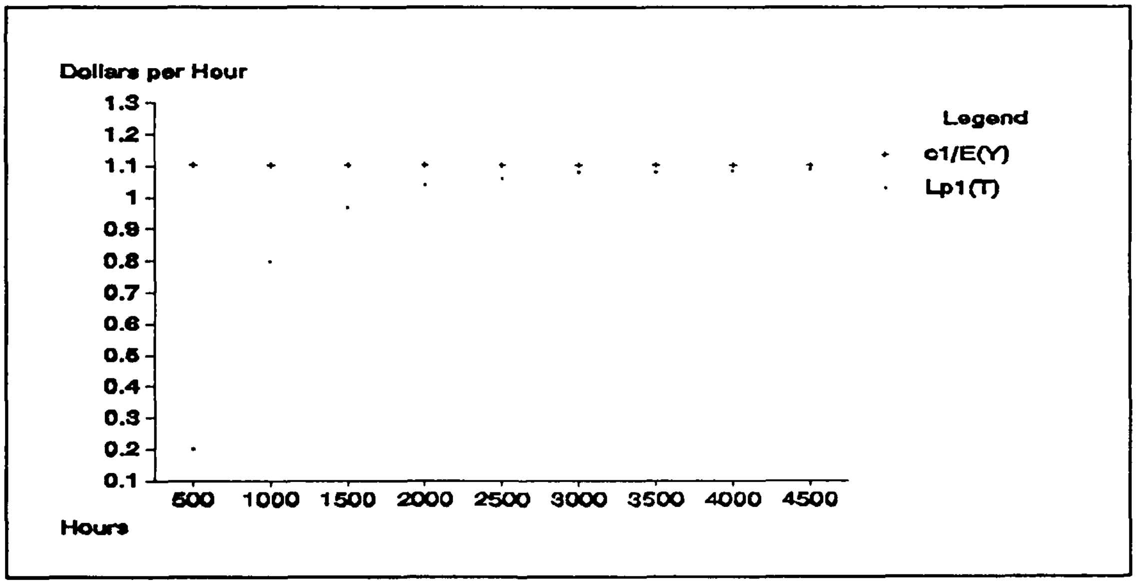

Figure 4. Values of $L_{p_1}(T)$ for k=500, p_1=.5.

cost is strictly increasing and T^* = k as expected from Theorem 2.2.b. since $.5 = p_1 \leq \exp(c_2/(c_2-c_1) + kr_0) =$

$\exp(-.1 + 500(.0000051)) \approx .9055$. A lower bound for T^* is $(c_2/c_1)EY = 91$.

We summarize the results of these cases in the Table 5 which follows.

Table 5. Summary of Examples

p_1	k	$(c_2/c_1)EY$	T^*	EY	$L_{p_1}(T)$
1.00	0	825	4146	9080	\$.036
.95	1000	694	4000	7629	\$.071
.95	500	544	4000	5979	\$.127
.50	1000	180	1000	1980	\$.100
.50	500	91	500	998	\$.200

Recall from Theorem 2.2.e. that $(c_2/c_1)EY$ is a lower bound for T^*. It appears that EY is an upper bound for T^*. Note that for p=.95 and k=1000, and p=.95 and k=500, when $T^* < \infty$ and $T^* \neq k$, we could have used the method in Example 1 p.90 of Barlow and Proschan (1965) to compute the optimal replacement age, then chosen the closest integer multiple of k to get the desired result.

Next, to observe asymptotic and monotonicity properties of $L_{p_1}(T)$, for k=1000, we compare $L_{p_1}(T)$ for

values of p_1 from .5 to 1.0 in Figure 5. For p_1 = .5, .7, and .9 we have T^* = k = 1000. Clearly $L_{p_1}(T^*)$ and $L_{p_1}(T)$ are decreasing in p_1. The upper bound is $L_0(T)$=1.1, and the sharp lower bound is $L_1(T)$. Horizontal asymptotes are labeled Lp_1(inf).

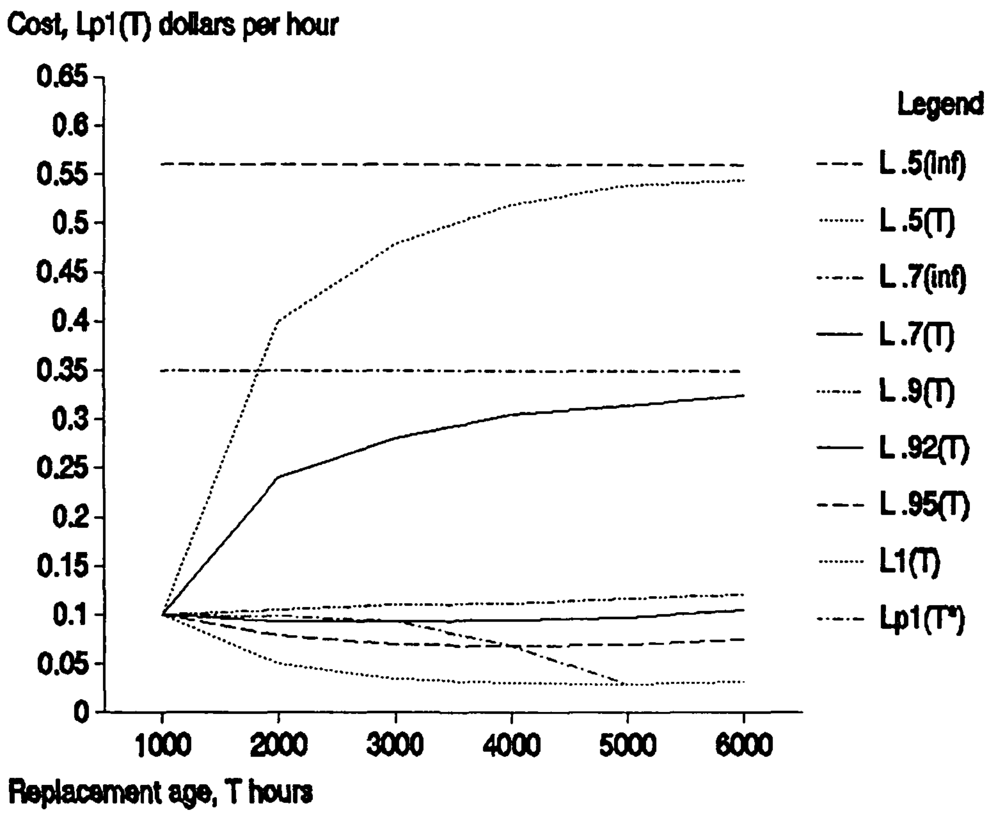

Figure 5. Comparison of $L_{p_1}(T)$ for k=1000.

3. MODEL 2: CUMULATIVE DAMAGE ASSESSMENT WITH TYPE I INSPECTION ERROR

3.1. Derman and Sacks Model and Stopping Lemma.

Derman and Sacks (1960) considered the problem of choosing an optimal replacement rule when the amount of deterioration of the equipment can be observed at each inspection time. The criterion for evaluating the effectiveness of the replacement rule is based on expected cost per unit of time per replacement cycle instead of long-run expected cost per unit of time. We will summarize their results and give an extension that incorporates the probability of a Type I inspection error.

Let Z be the lifelength of a device with distribution function G. Let c_1 be the cost of replacement after failure and c_2 be the cost of replacement before failure. At each inspection time t=1,2,... we observe the amount of deterioration X_t which occurred during the interval (t-1,t). Assume that $\{X_t\}$ is a sequence of i.i.d., nonnegative, random variables with common distribution function F. The equipment fails when the cumulative damage $S_n = \sum_{t=1}^{n} X_t$ exceeds a given threshold L.

We restrict ourselves to the class ζ of replacement

rules of the form $N=\min(\nu,[Z]+1)$, where $[Z]$ denotes the greatest integer less than Z, and ν is an optional stopping rule based on the cumulative damage. We wish to find a rule $N^* \in \zeta$ which minimizes the average cost per unit of time per replacement cycle. This is accomplished by using the following optimal optional stopping lemma. For the proof see Derman and Sacks (1960).

<u>Lemma 3.1</u>. Let $\{\mathscr{F}_n, n=1,2,\ldots\}$ be an increasing sequence of σ-fields of a sample space Ω. Let $\{Y_n\}$ be a sequence of random variables which are $\mathscr{F}_n$ measurable, and such that EY_n exists and is finite for all n. Let $\mathfrak{C}$ be the class of all optional stopping rules such that $EN < \infty$. If there exists an optional stopping rule N^* with

a. $EN^* < \infty$,

b. $E\{Y_n \mid \mathscr{F}_{n-1}\} \le Y_{n-1}$ when $n \le N^*(\omega)$

$\qquad\qquad \ge Y_{n-1}$ when $n > N^*(\omega)$

for almost all $\omega \in \Omega$; and if there is some $M < \infty$ such that

c. for all n, $E\{|Y_{n+1}-Y_n| \mid \mathscr{F}_n\} \le M$,

then $EY_{N^*} = \min\limits_{N\in\mathfrak{C}} EY_N$. Thus N^* is an optimal stopping rule.

<u>3.2. Application of Optimal Optional Stopping Lemma</u>.

In order to apply the lemma to the replacement problem, let

$$Y_n = \begin{cases} c_2/n \text{ if } n<Z \\ \dfrac{c_1}{[Z]+1} \text{ if } n\geq Z. \end{cases}$$

Let N be the number of time periods until a replacement under our replacement rule. Then we wish to choose N to minimize EY_N.

Derman and Sacks show that if we choose n^* to be the smallest n such that

$$1-F(L-S_{n-1}) \geq \frac{c_2}{(c_1-c_2)(n-1)},$$

then $N^* = \min(n^*-1, [Z]+1)$ is the optimal rule.

3.3. Extension of the Model for Type I Error.

We define $q_1(S_n)$ as the probability of mistakenly replacing a functioning unit after observing the cumulative damage S_n, because we conclude that the threshold has been exceeded when actually it has not been exceeded. Since we are more likely to make such a mistake as the cumulative damage nears the threshold value, it makes sense for $q_1(S_n)$ to be an increasing function of n. Then $p_1(S_n)=1-q_1(S_n)$ is the probability of concluding that the cumulative damage does not exceed the threshold when it actually does not exceed the threshold. On the other hand, a failed unit is replaced at inspection with probability one.

Define W to be the observed lifelength of the

equipment so that $P[W>i] = \overline{G}_i \prod_{j=0}^{i} p_1(S_j)$, for i=0,1,2,..., where $p_1(S_0)=1$. Let

$$Y_n = \begin{cases} c_2/n \text{ if } W>n \\ \sum_{i=1}^{[W]} \frac{c_2}{i} \prod_{j=0}^{i-1} p_1(S_j) q_1(S_i) + \frac{c_1}{[W]+1} \prod_{i=0}^{[W]} p_1(S_i) \text{ if } W \le n. \end{cases}$$

Since the Y_n are uniformly bounded, condition c. of the lemma holds. We see that if n-1 < W, then

$$\begin{aligned} E[Y_n | X_1,\dots,X_{n-1},p_1(S_1),\dots,p_1(S_{n-1})] &= \frac{c_2}{n} F(L-S_{n-1}) + \frac{c_1}{n}[1-F(L-S_{n-1})] \\ &= \frac{c_2}{n} + \frac{(c_1-c_2)}{n}[1-F(L-S_{n-1})], \end{aligned}$$

and if $n-1 \ge W$, then

$$E[Y_n | X_1,\dots,X_{n-1},p_1(S_1),\dots,p_1(S_{n-1})] = \sum_{i=1}^{[W]} \frac{c_2}{i} \prod_{j=0}^{i-1} p_1(S_j) q_1(S_i) + \frac{c_1}{[W]+1} \prod_{i=0}^{[W]} p_1(S_i).$$

Let $\mathcal{F}_n$ be the σ-field generated by $X_1,\dots,X_n,p_1(S_1),\dots,p_1(S_n)$. Then

$$E(Y_n | \mathcal{F}_{n-1}) \begin{cases} < \frac{c_2}{n-1} = Y_{n-1} \text{ if } n-1<W \text{ , } 1-F(L-S_{n-1}) < \frac{c_2}{(c_1-c_2)(n-1)}, \\ \ge \frac{c_2}{n-1} = Y_{n-1} \text{ if } n-1<W \text{ , } 1-F(L-S_{n-1}) \ge \frac{c_2}{(c_1-c_2)(n-1)}, \\ = Y_{n-1} \text{ if } n-1 \ge W. \end{cases}$$

Let n^* be as defined above. Then, for all $n \ge n^*$, we have $1-F(L-S_{n-1}) \ge c_2/(c_1-c_2)(n-1)$, since $1-F(L-S_{n-1})$ is

increasing in n and $c_2/(c_1-c_2)(n-1)$ is decreasing in n. Define $N^{**} = \min(n^{*}-1, [W]+1)$. Then routine computation shows that condition b. of the lemma is satisfied. Also $EN^{**} \le 1 + EW < \infty$; thus condition a. is satisfied. Similarly all replacement rules in ζ have finite expectations. Thus N^{**} is the optimal rule. Note that $N^{**} \le N^{*}$, since there will be more frequent replacements when there is a chance of inspection error.

4. MODEL 3: OPTIMAL INSPECTION SCHEDULE FOR A RR_2 DENSITY

4.1. Formulation of the Model.

Barlow, Hunter, and Proschan (1963) presented a model and algorithm to obtain an optimum inspection schedule with results based on the assumption that the system is deteriorating, specifically that the lifelength density function, f, is a Pólya frequency function of order 2 (PF_2). Before presenting the impact of Type II inspection error on this model in Chapter 5, this chapter presents results obtained under the assumption that the system is aging beneficially, specifically, that the lifelength density function f(x-y) is reverse rule 2 (RR_2).

Consider a system for which the lifelength T has known failure distribution F with finite mean μ_T. Suppose that system failure is discovered only by inspection and possibly only after some time has elapsed since failure. Assume that inspection takes negligible time, does not degrade the system, and system failure cannot occur during inspection. Each inspection costs c_1, so we do not wish to check too often. On the other hand, there is a cost c_2 per unit of time elapsed between system failure and detection of failure, so that we wish to check often enough to detect

failure quickly. Assume an inspection always follows failure. Upon detection of failure, inspection ends.

Our goal is to determine an inspection policy which minimizes the total expected cost resulting from both inspection and failure.

As shown in Chapter 4, Section 2.4 of Barlow and Proschan (1965), any random schedule may be improved upon by a nonrandom schedule. Thus define a checking procedure by an increasing sequence of nonnegative numbers $\{x_k\}_{k=0}^{\infty}$, where $x_0=0$ and the support of F is equal to $[0, \lim_{n\to\infty} x_n)$. For k=1,2,..., the kth check is performed at time x_k if the system has not failed by the (k-1)st check. If the system fails at time t, where $x_k < t \le x_{k+1}$, then the total cost incurred is $c_1(k+1) + c_2(x_{k+1}-t)$. Hence the expected cost is

$$C = \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} [c_1(k+1) + c_2(x_{k+1}-t)]\,dF(t)$$
$$= \sum_{k=0}^{\infty} [c_1(k+1) + c_2\, x_{k+1}][F(x_{k+1})-F(x_k)] - c_2\,\mu_T. \tag{4.1}$$

When F is a continuous life distribution with a finite mean, then an optimum checking schedule must exist, as proven in Barlow, Hunter, and Proschan (1963). Next they show how the optimum checking schedule may be obtained if the failure density, f(x), is PF_2. We will present analogous results when f(x-y) is RR_2. For convenience, the results from Barlow, Proschan, and Hunter (1963) are stated

here first without proofs.

4.2. Results for a PF_2 Density.

From Barlow and Proschan (1981) p. 76, we have the following definition for a Pólya frequency function of order 2.

Definition 4.1. A function h(x), $-\infty < x < \infty$, is PF_2 if

(a) $h(x) \geq 0$ for $-\infty < x < \infty$, and

(b)
$$\begin{vmatrix} h(x_1-y_1) & h(x_1-y_2) \\ h(x_2-y_1) & h(x_2-y_2) \end{vmatrix} \geq 0$$

for all $-\infty < x_1 < x_2 < \infty$ and $-\infty < y_1 < y_2 < \infty$. Equivalently,

(b′) ln h(x) is convex on $(-\infty,\infty)$, or

(b″) for fixed $\Delta > 0$, $h(x+\Delta)/h(x)$ is nonincreasing in x for $a \leq x \leq b$, where $a = \inf_{h(y)>0} y$, $b = \sup_{h(y)>0} y$.

Thus it can be seen from Definition 4.1 that log-concave functions and PF_2 functions are the same. We define a "strictly PF_2" function by using a strict inequality, >, in Definition 4.1 (b).

Assuming a failure density f, a necessary condition that a sequence $\{x_k\}$ be a minimum-cost checking procedure is that $\partial C/\partial x_k = C$ or all k. Hence using (4.1), we obtain for k=1,2,...,

$$\delta_k \equiv (x_{k+1}-x_k) = \frac{F(x_k)-F(x_{k-1})}{f(x_k)} - \frac{c_1}{c_2} . \tag{4.2}$$

When $f(x_k)=0$, $x_{k+1}-x_k=\infty$; thus no more checks are scheduled. The sequence is determined recursively once we choose x_1. The optimum checking policy is denoted $\{x_k^*\}$.

<u>Theorem 4.1</u>. If the failure density f is PF_2, and $f(x) > 0$ for $x > 0$, the optimum checking intervals are nonincreasing.

<u>Theorem 4.2</u>. Let f be strictly PF_2 with $f(x) > 0$ for $x > 0$ If $x_1 > x_1^*$, then $\delta_n > \delta_{n-1}$ for some positive integer n. If $x_1 < x_1^*$, then $\delta_n < 0$ for some positive integer n.

<u>Algorithm 4.1</u>.

1. To balance the cost of a single check against the expected cost of undetected failure occurring before the first check, choose x_1 to satisfy:

$$c_1 = c_2\int_0^{x_1}(x_1-t)\,dF(t) = c_2\int_0^{x_1}F(t)\,dt.$$

2. Compute $x_2, x_3, \ldots$ from (4.2).

3. If $\delta_k > \delta_{k-1}$, reduce x_1 and repeat. If $\delta_k < 0$, increase x_1 and repeat.

Intuitively these results are not surprising. Since a

PF_2 density has an increasing failure rate distribution (IFR), we need to check more and more often as the system failure rate increases. Similarly, we will show that we need to check less and less frequently if the system failure rate decreases. Specifically, we will show that optimum checking intervals are nondecreasing if the failure density f(x-y) is RR_2. Then we will present a procedure to derive the optimum schedule. Finally note the decreasing failure rate (DFR) Weibull, gamma, and exponential distributions all satisfy these conditions; thus for these distributions, nondecreasing checking intervals are optimal.

4.3. Results for a RR_2 Density.

From Karlin (1968) p. 12, we have the following definition for a function which is reverse rule of order 2.

Definition 4.2. Let A and B be subsets of the real line. A function K(x,y) on A×B is said to be reverse rule of order 2 (RR_2) if

$$\begin{vmatrix} K(x_1,y_1) & K(x_1,y_2) \\ K(x_2,y_1) & K(x_2,y_2) \end{vmatrix} \le 0$$

for all $x_1 < x_2$ in A and $y_1 < y_2$ in B.

We are interested in the special case K(x,y) = h(x-y). From the above definition, it is easy to verify for a RR_2

function h(x-y), that h(x) is log-convex. In other words, by reversing the inequality in Definition 4.1 (b) we obtain the definition of a log-convex function. If strict inequality, <, holds, then we have a "strictly log-convex" function.

Note that a log-convex function is not necessarily a density function. But if a density function f is log-convex with support on $[0,\infty)$, then the corresponding distribution F is DFR. The proof is analogous to that for corresponding conditions on a PF_2 density in Barlow and Proschan (1981) p.77.

Part (b) of the definition gives us a very simple way to show when the Weibull and gamma distributions are DFR. The following example is exercise 10 on page 79 in Barlow and Proschan (1981).

Example 4.1. Let g be the gamma density:

$$g(t;\lambda,\alpha) = \frac{\lambda^{\alpha}\, t^{\alpha-1}}{\Gamma(\alpha)}\exp(-\lambda t) \quad \text{for } t\geq 0,\ \lambda>0,\ \alpha>0.$$

Then $\ln g(t;\lambda,\alpha) = \ln[\lambda^{\alpha}/\Gamma(\alpha)] + (\alpha-1)\ln t - \lambda t$.

Differentiating twice we get

$$\frac{d^2}{dt^2}\ln g(t;\lambda,\alpha) = -(\alpha-1)t^{-2} \begin{cases} > 0 & \text{for } 0<\alpha<1 \\ < 0 & \text{for } \alpha>1. \end{cases}$$

Thus g is log-convex and G is DFR for $0<\alpha\leq 1$, while g is PF_2 (log-concave) and G is IFR for $\alpha\geq 1$.

We need the following lemmas to prove that optimal checking intervals are nondecreasing.

Lemma 4.1. A density f is log-convex if and only if, for all Δ, $[F(x+\Delta)-F(x)]/f(x)$ is nondecreasing in x.

The proof is analogous to that of Theorem 1, Appendix 1 of Barlow, Hunter, and Proschan (1963) where f is PF_2.

Equivalent to Lemma 4.1, a density f is log-convex if and only if for all $\Delta \geq 0$, $[F(x+\Delta)-F(x)]/f(x)$ is nondecreasing in x, and for all $\Delta \geq 0$, $[F(x)-F(x-\Delta)]/f(x)$ is nonincreasing in x.

Lemma 4.2. If f is log-convex, where $x < y$, $a \geq 1$, and $\Delta \geq 0$, then, $a\left[\frac{F(y)-F(y-\Delta)}{f(y)}\right] < \frac{F(x)-F(x-a\Delta)}{f(x)}$.

The proof is analogous to the proof of Theorem 3, Appendix 1 of Barlow, Hunter, and Proschan (1963) where f is PF_2.

Theorem 4.3. If the failure density f is log-convex, and $f(x) > 0$ for $x > 0$, then optimum checking intervals are nondecreasing.

Proof. Let $\{x_k^*\}$ denote the optimum checking policy and $\{x_k\}$

another checking policy obtained from (4.2). Assume for some k and $a > 1$ that $\delta_{k-1}/\delta_{k-2} = 1/a < 1$. We will show that this implies $\delta_k/\delta_{k-1} \le 1/a$.

Note that

$$a\delta_k - \delta_{k-1} = a\left[\frac{F(x_k)-F(x_{k-1})}{f(x_k)}\right] - \left[\frac{F(x_{k-1})-F(x_{k-2})}{f(x_{k-1})}\right] + \frac{c_1}{c_2}(1-a).$$

The right hand side is negative by Lemma 4.2 and the assumption that $a\delta_{k-1} = \delta_{k-2}$. Hence $a\delta_{k-1} \le \delta_{k-2}$ implies that $a\delta_k \le \delta_{k-1}$. Thus we conclude that if for any k, $\delta_k < \delta_{k-1}$, then $\delta_n \to 0$ geometrically fast from some point on as $n \to \infty$.

Next we show $\delta_k^* < \delta_{k-1}^*$ for some k contradicts the fact $\{x_k^*\}$ is an optimum policy. Notice x_{n+1}^* is the optimum first checking point for the conditional density $f(t+x_n^*)/\overline{F}(x_n^*)$. Expected cost for the optimum first check is then,

$$c_1 + c_2\int_{x_n^*}^{x_{n+1}^*}(x_{n+1}^*-t)\,\frac{f(t)\,dt}{\overline{F}(x_n^*)} = c_1 + c_2\int_0^{\delta_n}(\delta_n-t)\,\frac{f(t+x_n^*)\,dt}{\overline{F}(x_n^*)} \to c_1$$

since $\delta_n \to 0$ as $n \to \infty$. This is a contradiction because the expected cost for the optimum first check using the density $f(t+x_n^*)/\overline{F}(x_n^*)$ is greater than c_1 as follows since $f' < 0$:

$$c_1 + c_2\int_{x_n^*}^{x_{n+1}^*}(x_{n+1}^*-t)\,\frac{f(t)\,dt}{\overline{F}(x_n^*)} \ge c_1 + c_2\frac{f(x_{n+1})}{\overline{F}(x_n^*)}\int_{x_n^*}^{x_{n+1}^*}(x_{n+1}^*-t)\,dt > c_1.$$

Hence optimum checking intervals are nondecreasing. ‖

Now since the exponential density is both log-concave and log-convex, applying Theorems 4.1 and 4.3 we obtain the following result, which is stated without proof in Barlow, Proschan, and Hunter (1963).

<u>Corollary 4.1</u>. If $f(x) = \lambda\exp(-\lambda x)$, $x \geq 0$, $\lambda > 0$, then optimum checking intervals are a constant $\delta_i = m$, where m is the unique solution of $\exp(\lambda m) - 1 = \lambda(c_1/c_2 + m)$.

To obtain an algorithm for computing an optimum checking sequence when f is log-convex we need the following lemma.

<u>Lemma 4.3</u>. Assume f is log-convex. Let $\{x_k\}$ be generated by (4.2) with $\delta_k > 0$ for $k=1,2,\ldots,N$ where $\delta_k = x_{k+1} - x_k$.

Then $\dfrac{d\delta_k}{dx_1} \leq \dfrac{f(x_1)}{f(x_k)} \dfrac{d\delta_1}{dx_1}$.

The proof is by induction, analogous to the proof of Lemma 1 in Barlow, Hunter, and Proschan (1963).

<u>Theorem 4.4</u>. Let f be strictly log-convex with $f(x) > 0$ for $x > 0$. If $x_1 < x_1^*$, then $\delta_n < 0$ for some positive integer n.

The proof is analogous to the proof of Theorem 6 in Barlow, Hunter, and Proschan (1963).

Algorithm 4.2.

1. Choose x_1 as in Algorithm 4.1.

2. Compute x_2, x_3,... recursively from (4.2).

3. If any $\delta_k < 0$, increase x_1 and repeat above. If $\delta_k < \delta_{k-1}$ for some k, reduce x_1 and repeat above.

4. Repeat to desired level of accuracy.

Example 4.2. Suppose the time to failure has a Weibull density $f(t) = \alpha\lambda^{\alpha}t^{\alpha-1}\exp[-(\lambda t)^{\alpha}]$ with $\alpha=.5$, $\lambda=.1$, $c_1=10$, and $c_2=1$. Thus f is DFR with a mean life of 20 and cost per inspection is 10 times the cost per unit of time for undetected failure. By step 1 we estimate $x_1=17.676$ which gives us all $\delta_k \geq 0$. The next estimate $x_1=8$ results in $\delta_6 < 0$. Repeating step 3 eventually yields $8.546 < x_1^* < 8.547$. Checking intervals obtained for $x_1=17.676$, $x_1=8$, and $x_1=8.547$ are contained in Tables 6 and 7 below.

Using (4.1) to compute the optimum cost we get C=27.746 when $x_1^*=8.547$. To see the impact on cost if we do not follow the optimum sequence, suppose we just change $x_2^*=26.663$ to $x_2=20$. Then the cost computed from (4.1) is 28.314. The cost obtained by using the sequence starting with $x_1=17.676$ through $x_5=9.957\text{x}10^{46}$ is C=45.764.

Table 6. Optimum Checking Sequences: x_1=17.676, x_1=8.000

k	x_k	δ_k	x_k	δ_k
1	17.676	63.899	8.000	15.866
2	81.575	195.793	23.866	18.313
3	1.340×10^3	1.058×10^3	42.179	17.250
4	1.250×10^5	1.242×10^5	59.429	12.828
5	9.957×10^{46}	9.957×10^{46}	72.258	5.289
6	9.957×10^{46}	0	77.547	-4.348

Table 7. Optimum Checking Sequence for x_1=8.547

k	x_k	δ_k	k	x_k	δ_k
1	8.547	18.116	16	593.556	55.552
2	26.663	23.660	17	649.108	58.091
3	50.323	27.740	18	707.199	61.149
4	78.063	31.041	19	768.348	64.973
5	109.105	33.853	20	833.321	69.926
6	142.958	36.328	21	903.247	76.564
7	179.286	38.558	22	979.811	85.779
8	217.845	40.606	23	1.07×10^3	99.089
9	258.451	42.519	24	1.17×10^3	119.249
10	300.970	44.335	25	1.28×10^3	151.883
11	345.305	46.090	26	1.43×10^3	210.089
12	391.395	47.823	27	1.65×10^3	331.771
13	439.218	49.576	28	1.98×10^3	674.578
14	489.794	51.400	29	2.65×10^3	2.67×10^3
15	540.194	53.362	30	5.32×10^3	4.09×10^5

Next in Chapter 5, we modify the model by introducing the possibility of a Type II inspection error.

5. MODEL 4: OPTIMAL INSPECTION SCHEDULE WITH TYPE II INSPECTION ERROR

5.1. Formulation of the Model.

Consider a system for which the lifelength T has known failure distribution F with finite mean μ_T. Model 4 is identical to Model 3 in the last chapter except that now we let $p_2 < 1$ be the probability that failure is detected upon inspection and $q_2 = 1 - p_2$ be the probability that we fail to detect a system failure at an inspection. If the system fails at time t, where $x_k < t \le x_{k+1}$, then the probability the failure is detected at time x_{k+i} is $p_2 q_2^{i-1}$, and the cost incurred is then $c_1(k+i) + c_2(x_{k+i}-t)$. Hence the expected cost is

$$C_{p_2} = \sum_{i=1}^{\infty} p_2 q_2^{i-1} \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} [c_1(k+i) + c_2(x_{k+i}-t)]dF(t)$$
$$= -c_2\mu_T + \sum_{k=0}^{\infty} [c_1 k + \frac{c_1}{p_2} + c_2 \sum_{i=1}^{\infty} p_2 q_2^{i-1} x_{k+i}][F(x_{k+1}) - F(x_k)]. \quad (5.1)$$

5.2. Existence of an Optimal Solution.

The proof of the existence of an optimal checking sequence follows the proof for $p_2=1$ in Barlow, Hunter, and Proschan (1963) with some modifications. Given that a failure is detected at the jth inspection following failure

with probability $p_2=1$ the expected cost is

$$C_j = \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} [c_1(k+j) + c_2(x_{k+j}-t)]dF(t).$$

It can be shown that an upper bound for the expected cost is $c_1 j + 2\sqrt{c_1 c_2 \alpha j}$, following the proof of Lemma, p.1080, Barlow, Hunter, and Proschan (1963).

Next we can show that $L_j(x)$, the infimum of expected cost over all policies given that failure is detected at the jth inspection and given that the component is of initial age x, is continuous. The proof follows the proof of Theorem 1, p. 1081, Barlow, Hunter, and Proschan (1963). Then the infimum of expected cost over all policies, $p_2 \leq 1$, given the component is of initial age x is

$$L_{p_2}(x) = \sum_{j=1}^{\infty} p_2 q_2^{j-1} L_j(x).$$

By the Weierstrass M-test, $\sum_{j=1}^{\infty} p_2 q_2^{j-1} L_j(x)$ converges uniformly to $L_{p_2}(x)$, and since each $p_2 q_2^{j-1} L_j(x)$ is continuous, then $L_{p_2}(x)$ is continuous. The remainder of the proof follows the proof of Theorem 1, p. 1081, Barlow, Hunter, and Proschan (1963).

A necessary condition for a sequence $\{x_k\}$ to be a minimum cost checking procedure is that $\partial C_{p_2}/\partial x_k = 0$ for all k. Hence using (5.1), we obtain

$$\delta_k = x_{k+1} - x_k = N_k - q_2 N_{k+1} - \frac{c_1}{c_2}, \qquad (5.2)$$

$$\text{where } N_k \equiv \sum_{i=1}^{k} \frac{[F(x_i) - F(x_{i-1})]q_2^{k-i}}{f(x_k)}.$$

The sequence is determined recursively once we choose x_1.

5.3. Schedule for Systems with Finite Lifelength.

Realistically, no system lives forever without repair or replacement. If it is known that a system lifelength must be less than $T < \infty$, we can give necessary and sufficient conditions on the failure distribution F to ensure that an optimal checking policy consists of a single check at time T. Maintaining the same assumptions as given in the previous model, we assume a check always follows a failure.

Theorem 5.1. Let $F(t)=1$ for $t \geq T$. If

$$F(t) \leq \frac{1}{p_2\left[1 + \frac{c_2}{c_1}(T-t)\right]} \qquad (5.3)$$

for $0 \leq t \leq T$, then the optimum policy consists of a single check at time T. Conversely, if

$$F(t) > \frac{1}{p_2\left[1 + \frac{c_2}{c_1}(T-t)\right]} \qquad (5.4)$$

for some $0 \leq t \leq T$, then the optimum policy will require in

addition to the check at time T one or more checks before time T.

Proof. If a single check is performed at time T, the expected cost is

$$D_1 = c_1 + c_2\int_0^T (T-t)\,dF(t) = c_1 + c_2\int_0^T F(t)\,dt.$$

If an additional check is performed at time x, then the expected cost is

$$D_2(x) = p_2\int_0^x [c_1+c_2(x-t)]\,dF(t) + q_2\int_0^x [2c_1+c_2(T-t)]\,dF(t)$$
$$+ \int_x^T [2c_1+c_2(T-t)]\,dF(t).$$

Thus a single check at time T is preferable if for all $0 \le x \le T$, $D_2(x) - D_1 \ge 0$; or equivalently,

$D_2(x) - D_1 = c_1[1-p_2F(x)] - c_2(T-x)p_2F(x) \ge 0,$

which gives us (5.3).

Now assuming a single check at T is preferred to checks at x and T, we show that a single check at T is preferable to a check at time T preceded by two checks at, say, times x_1 and x_2. Define $\underline{x}_k=(x_1,\ldots,x_k)$. Let $D_3(\underline{x}_2)$ be the cost of checking at times x_1, x_2, and T. Given that a check has occurred at time x_1, a single check before T, namely at time x_2, is warranted if and only if $D_3(\underline{x}_2)-D_2(\underline{x}_1)$

< 0, or equivalently,

$$D_3(\underline{x}_2)-D_2(\underline{x}_1)=c_1[1-p_2F(x_2)-p_2q_2F(x_1)]-c_2p_2(T-x_2)[F(x_2)-p_2F(x_1)]< 0.$$

This gives us

$$\frac{F(x_2)-p_2F(x_1)}{1-p_2F(x_1)} > \frac{1}{p_2[1+(T-x_2)c_2/c_1]}$$

for some $x_1 \le x_2 \le T$. But the above implies $F(x_2)> 1/p_2[1+(T-x_2)c_2/c_1]$ since $[1-p_2F(x_1)]F(x_2)> F(x_2)-p_2F(x_1)$. But this contradicts our assumption that $F(x)\le 1/p_2[1+(T-x)c_2/c_1]$ for $0 \le x \le T$.

Similarly it is easy to show that (5.3) implies a single check at time T is preferable to k checks before T at times x_1, x_2, ... x_k. The expected cost of performing k+1 checks at times x_1, x_2, ..., x_k, T is

$$D_{k+1}(\underline{x}_k) = D_k(\underline{x}_{k-1}) + c_1(1-M_{k+1}) - c_2p_2(T-x_k)[F(x_k)-M_k]$$
$$\text{where } M_1\equiv 0,\ M_k\equiv p_2\sum_{i=1}^{k-1}F(x_{k-i})q_2^{i-1} \text{ for } k=2,3,... \tag{5.5}$$

Given that checks have occurred at times x_1, x_2, ..., x_{k-1}, an additional check at some time x_k, where $x_{k-1}< x_k< T$, is warranted if and only if $D_{k+1}(\underline{x}_k)-D_k(\underline{x}_{k-1}) < 0$, or equivalently,

$$\frac{F(x_k)-M_k}{1-M_k} > \frac{1}{p_2[1+(T-x_k)c_2/c_1]}.$$

But the above inequality and the fact that $F(x_{k-1}) > M_k$ imply

$$\frac{F(x_k) - F(x_{k-1})}{1 - F(x_{k-1})} > \frac{1}{p_2[1+(T-x_k)c_2/c_1]},$$

which in turn implies $F(x_k) > 1/p_2[1+(T-x_k)c_2/c_1]$, contradicting our initial assumption that a single check at T is preferred. Conversely if (5.3) is not true for some x in [0,T], then checks at times x and T will yield a lower expected cost than does a single check at T. Thus the optimum checking policy will require one or more checks before T. ‖

Inequality (5.4) suggests that we would prefer to conduct more checks than just a single check at T if the cost per inspection, c_1, is sufficiently small compared to the cost of failure, c_2, or if the probability of detecting a failure, p_2, is sufficiently large.

We can compute a checking sequence, $x_1, x_2, \ldots, x_n=T$ and the expected cost, $D_n(\underline{x}_{n-1})$, recursively from (5.2) and (5.5) once we have chosen x_1. If our choice of x_1 is greater than the unknown optimal value x_1^*, then the checking sequence computed from (5.2) will not contain an inspection time $x_n=T$, but rather will contain an x_n exceeding T for some n. On the other hand if our choice of x_1 is less than the optimal x_1^*, the checking sequence computed from (5.2) will have $x_k < T$ for all k, and for

some n we will obtain $x_{n+1} < x_n$. A reasonable first guess for x_1 would be the same as in Algorithm 4.1 where we balance the expected cost of a failure occurring before x_1 against the cost of a single inspection at x_1.

Example 5.1. Suppose it is known that a system will not function after time T=10. Assume the life distribution of the system can be represented by an exponential distribution truncated at T=10 with parameters λ=.1 and cost factors c_1=1 and c_2=10. Then

$$\begin{aligned} f(x) &= .1e^{-.1x} / \int_0^{10} .1e^{-.1t} dt \quad \text{for } 0 \le x \le 10 \\ &= .1e^{-.1x} / (1-e^{-1}). \end{aligned}$$

Let the probability of failure detection be p_2=.9. Our first guess for x_1 is 1.146 which is too large. A second guess of x_1=1.0 is too small. Continuing to choose values for x_1 halfway between a value which is too small and a value which is too large, we can obtain an optimal value for x_1 to whatever degree of accuracy is required to make the sequence end at T=10. The sequences obtained using x_1=1.146, x_1=1.0, and x_1=1.0933 and their associated costs are given in Table 8 on the next page.

Using Theorem 5.1 we can show when p_2=.9 and x_1=1.09330 that 17 inspections are preferred to 16, but 18 inspections are not preferred to 17. So the optimal checking schedule requires 17 inspections, resulting in an expected cost of

Table 8. Checking Schedules $\{x_i\}$ and Costs $\{D_i(x)\}$ for a Truncated Exponential λ=.1, T=10, c_1=1, c_2=10, for p_2=.9

i	x_i	D_i	x_i	D_i	x_i	D_i
1	1.146	59.198	1.000	59.198	1.09330	59.198
2	2.142	46.394	1.852	47.868	2.03695	46.917
3	3.109	36.435	2.661	38.937	2.94612	37.314
4	4.044	28.790	3.423	31.960	3.81743	29.889
5	4.944	23.017	4.134	26.563	4.64732	24.223
6	5.805	18.744	4.790	22.435	5.43207	19.967
7	6.624	15.664	5.387	19.317	6.16783	16.829
8	7.397	13.517	5.921	16.995	6.85067	14.568
9	8.120	12.091	6.389	15.295	7.47663	12.984
10	8.789	11.210	6.787	14.075	8.04179	11.913
11	9.400	10.730	7.112	13.226	8.54233	11.222
12	9.949	10.532	7.361	12.660	8.97461	10.803
13	10.432	10.522	7.532	12.314	9.33526	10.570
14	10.846	10.523	7.623	12.144	9.62127	10.457
15			7.633	12.124	9.83006	10.413
16			7.561	12.246	9.95956	10.402
17			7.407	12.520	10.00828	10.40197

10.40197 versus the cost of 59.198 if only a single inspection is performed at T=10. Under the optimal schedule for p_2=1, we obtain x_1=1.0859344 and only 15 inspections are required with an expected cost of 9.436. Thus when p_2<1 our initial inspection time, x_1=1.09330, is later than when p_2=1, x_1=1.0859344. But we must inspect more frequently when p_2<1, resulting in a higher expected cost. The optimal inspection schedule and cost when p_2=1 are given in Table 9 on the next page.

Table 9. Checking Schedules $\{x_i\}$ and Costs $\{D_i(x)\}$ for a Truncated Exponential $\lambda=.1$, $T=10$, $c_1=1$, $c_2=10$, for $p_2=1$

i	x_i	D_i	i	x_i	D_i
1	1.0859344	59.198	9	7.9760442	11.137
2	2.130250	45.524	10	8.5363479	10.315
3	3.1369001	35.121	11	9.0126459	9.839
4	4.0928928	27.317	12	9.4004691	9.590
5	4.9960732	21.559	13	9.8957602	9.479
6	5.8412965	17.397	14	9.9976199	9.442
7	6.6232679	14.462	15	10.0000002	9.436
8	7.336261	12.455	16		

5.4. Optimal Checking Intervals for an Exponential Density when $0 < p_2 < 1$.

Due to the memoryless property of the exponential distribution, one would expect optimal checking intervals to be constant. But Sengupta (1982) showed that the first checking interval is longer, solving the problem as a continuous-time Markov decision process. Denote the first checking interval by $\delta_0=x_1$ and set $\delta_i=m$ for $i=1,2,\ldots,$ then we can solve for m from equation (5.1) and $\partial C_{p_2}/\partial x_1 = 0$ and $\partial C_{p_2}/\partial x_2 = 0$. Let $F(t) = 1 - \exp(-\lambda t)$; then

$$\begin{aligned} \lambda m &= -\ln(q_2 + p_2 e^{-\lambda x_1}) \\ \lambda x_1 &= \ln\left[1 + \left(\frac{c_1}{c_2} + m\right)\frac{\lambda}{p_2}\right]. \end{aligned} \tag{5.6}$$

Solving for m, we get

$$\lambda m = \ln\left[\frac{c_1}{c_2} + m + \frac{p_2}{\lambda}\right] - \ln\left[q_2\frac{c_1}{c_2} + q_2 m + \frac{p_2}{\lambda}\right]. \tag{5.7}$$

Thus our optimal checking sequence is x_1, x_1+m, x_1+2m, ... and the expected cost is

$$C_{p_2} = -\frac{c_2}{\lambda} + c_2(x_1-m) + (c_1+c_2m)\left[\frac{1}{p_2} + e^{-\lambda x_1}\left[\frac{e^{\lambda m}}{e^{\lambda m}-1}\right]\right]. \tag{5.8}$$

<u>Example 5.2</u>. Let F be an exponential distribution with $\lambda=.01$. Assume the probability of detecting failure at inspection is $p_2=.9$, the cost per unit of time for undetected failure is $c_2=1$ and the cost per inspection is $c_1=10$. Then using (5.6) and (5.7) we obtain $x_1=41.874$ and $m=36.805$. The expected cost from (5.8) is $C_{p_2} = 57.075$.

Table 10 below compares results for different values of p_2 when $\lambda=.01$, $c_1=10$, and $c_2=1$. Thus the smaller the probability of detecting a failure, the longer we wait to perform the first inspection, but then the remaining inspection intervals $\delta_i=m$ are shorter for smaller p_2. This reflects the lack of confidence in the ability to detect a failure, and as p_2 decreases, the expected cost C_{p_2} increases. These relationships are verified by finding the derivative with respect to p_2 of C_{p_2}, x_1, and m.

Table 10. Optimal Schedules and Costs for an Exponential Density $\lambda=.01$, $c_1=10$, $c_2=1$

p_2	x_1	m	Cp_2
1.0	41.622	41.622	51.622
.9	41.874	36.805	57.075
.8	42.630	32.526	63.262
.7	43.936	28.619	70.487
.6	45.913	24.961	79.220
.5	48.799	21.452	90.251
.4	53.056	17.995	105.048
.3	59.663	14.480	126.781
.2	71.141	10.737	164.091
.1	97.138	6.416	254.881

5.5. Optimal Checking Intervals for a PF_2 Density when $0 < p_2 < 1$.

For $p_2=1$, it is proven in Theorem 5, p. 1088 of Barlow, Hunter, and Proschan (1963) that optimal checking intervals are nonincreasing if the failure density f is PF_2.

For $0 < p_2 < 1$, the same conclusion, that optimal intervals are nonincreasing, is proven here when the failure density is PF_2. Parmigiani (1990d) formulated a model incorporating both Type I and Type II inspection errors. A Type I error occurs when an inspector mistakenly says that a system has failed when it is in fact functioning. A Type II error occurs when an inspection fails to detect a malfunctioning system. Parmigiani

(1990d) found that the problem of minimizing the long-run expected cost depends on the probability of a Type I error only as a function of the constant cost factors. Thus for simplicity of notation, we will use our Model 4, which uses only Type II error. First we present monotonicity properties of the model in Section 5.5.1. In Section 5.5.2, we prove that when the failure density is PF_2, optimal inspection intervals are nonincreasing. In Section 5.5.3, we obtain an algorithm for computing optimal intervals. Finally, we derive majorization results for an optimal inspection policy in Section 5.5.4.

Recall that in Model 4 a system has lifelength T with known failure distribution F with finite mean μ_T. The probability of detecting the failure of the system upon inspection is a constant p_2. We denoted a sequence of inspection times by $\{x_k\}$ for $k=0,1,...$ and $x_0 \equiv 0$. Failure of the system can only be detected at one of these inspection times. Let S be the time when failure is detected. Then

$$\begin{aligned} P(S=x_k) &= P(x_{k-1}<S\le x_k) \\ &= \sum_{i=0}^{k-1} p_2 q_2^{k-1-i} P(x_i<T\le x_{i+1}) \\ &= \sum_{i=0}^{k-1} p_2 q_2^{k-1-i}[F(x_{i+1})-F(x_i)]. \end{aligned} \tag{5.9}$$

Denote the failure distribution of S by G. Then

$$G(x_k) = P(S\le x_k) = \sum_{j=1}^{k}\sum_{i=0}^{j-1} p_2 q_2^{j-1-i}[F(x_{i+1})-F(x_i)]. \tag{5.10}$$

5.5.1 Monotonicity Properties.

It is reasonable to conjecture that as the probability of detecting a failure decreases, then the expected time to detect the failure will increase. We show that the expected time of failure detection, μ_S, is a decreasing function of p_2 as follows.

$$\mu_S = E(S) = \sum_{k=0}^{\infty}\sum_{i=1}^{\infty} p_2 q_2^{i-1} x_{k+i}[F(x_{k+1}) - F(x_k)].$$

$$\frac{d\mu_S}{dp_2} = \sum_{k=0}^{\infty}\sum_{i=1}^{\infty} x_{k+i}[F(x_{k+1}) - F(x_k)] q_2^{i-2}(1 - p_2 i).$$

By definition $x_{k+i} = x_k + \delta_k + \delta_{k+1} + \ldots + \delta_{k+i-1}$.

Let $\delta_m = \max\limits_{k \le j \le k+i-1} \delta_j$. Then $x_{k+i} < x_k + i\delta_m$.

Applying this to $d\mu_S/dp_2$ we get

$$\frac{d\mu_S}{dp_2} < \sum_{i=1}^{\infty} q_2^{i-2}(1 - p_2 i) \sum_{k=0}^{\infty} (x_k + i\delta_m)[F(x_{k+1}) - F(x_k)]$$
$$= \sum_{k=0}^{\infty} [F(x_{k+1}) - F(x_k)] \left(-\frac{\delta_m}{p_2^2}\right) < 0.$$

By a similar proof we can show that C_{p_2} is a decreasing function of p_2. Again it is reasonable that the long-run expected cost would decrease as the expected time to detect the failure decreases.

Another way to view the impact of imperfect inspection on the long-run expected cost C_{p_2} is that, C_{p_2} increases as

the difference between the expected time to failure-detection, μ_S, and the expected time to failure, μ_T, increases. This can be seen by writing

$$C_{p_2} = c_1 \sum_{k=0}^{\infty} k[F(x_{k+1}) - F(x_k)] + \frac{c_1}{p_2} + c_2(\mu_S - \mu_T).$$

5.5.2. Optimal Intervals are Nonincreasing for a PF_2 Failure Density.

When p_2=1, Barlow, Hunter, and Proschan (1963) proved intervals are nonincreasing for a PF_2 density. The proof by contradiction showed that if inspection intervals began increasing at some point in time, then they continued to increase geometrically, resulting in the expected cost also increasing to infinity. But when $0 < p_2 < 1$, an increase in inspection intervals at some point in time does not result in intervals increasing to infinity. However, we will prove here that the intervals between expected detection times, conditional on time of failure, increase geometrically and this is sufficient to provide the proof by contradiction.

Given an inspection sequence $\{x_k\}$, the expected time of detection of failure, S, conditioned on the time of failure, T, occurring in an particular interval is

$$z_k \equiv E(S | x_{k-1} < T \le x_k) = \sum_{i=0}^{\infty} p_2 q_2^i x_{k+i} \quad k=1,2,\ldots \tag{5.11}$$

The inspection times, x_k, and the conditional, expected detection times, z_k, are related by $z_k = p_2 x_k + q_2 z_{k+1}$; equivalently,

$$x_k = \frac{1}{p_2} z_k - \frac{q_2}{p_2} z_{k+1}. \tag{5.12}$$

The optimal inspection sequence $\{x_k\}$ minimizes the expected cost function (5.1) which by applying (5.11) can be written as follows

$$C_{p_2} = -c_2 \mu_T + \sum_{k=0}^{\infty} [c_1 k + \frac{c_1}{p_2} + c_2 z_{k+1}][F(x_{k+1}) - F(x_k)]. \tag{5.13}$$

Furthermore applying (5.12) to the optimal inspection intervals δ_k in (5.2) we obtain optimal intervals between conditional, expected detection times for k = 1,2,...

$$\gamma_k \equiv z_{k+1} - z_k = \sum_{i=1}^{k} p_2 q_2^{k-i} \frac{[F(x_i) - F(x_{i-1})]}{f(x_k)} - \frac{c_1}{c_2}. \tag{5.14}$$

Iteratively substituting the kth condition in the (k+1)st gives

$$\gamma_1 = p_2 \frac{F(x_1)}{f(x_1)} - \frac{c_1}{c_2}, \quad \text{and for } k=2,3,\ldots$$
$$\gamma_k = p_2 \frac{F(x_k) - F(x_{k-1})}{f(x_k)} + q_2 \frac{f(x_{k-1})}{f(x_k)} \left[\gamma_{k-1} + \frac{c_1}{c_2} \right] - \frac{c_1}{c_2} \tag{5.15}$$

Then from (5.12) and (5.14) we can write

$$\gamma_k = p_2 \delta_k + q_2 \gamma_{k+1} \qquad k=1,2,\ldots \tag{5.16}$$

To prove that optimal inspection intervals, δ_k, are nonincreasing for a PF_2 density, we need the following facts and lemmas. We assume that f is PF_2 and f′ exists everywhere. Let m denote the mode of f which is unique by Barlow and Proschan (1965) p.230.

Clearly when f is a PF_2 density and f′ exists everywhere, then $f^2(x) \geq f'(x)F(x)$. As a consequence, we have

$$f'(x) > 0 \Rightarrow f(x)/f'(x) \geq F(x)/f(x)$$
$$f'(x) < 0 \Rightarrow f(x)/f'(x) \leq F(x)/f(x).$$

Furthermore, we know when f is a PF_2 density and f′ exists everywhere, if x < y, then $f'(x)/f(x) \geq f'(y)/f(y)$. As a consequence of this fact, if $x_1 < x_2 < m$ (mode of f), or $m < x_1 < x_2$ then, $f(x_2)/f'(x_2) \geq f(x_1)/f'(x_1)$. The inequality is reversed when $x_1 < m$ and $x_2 > m$.

Lemma 5.1. Let $\{x_k\}$ be an inspection sequence generated by (5.2). Then $f_k/f_{k-1} > q_2^2$ for k=2,3,... .

Proof. By definition

$$\delta_k = \left[\frac{F_{k-1}-p_2F_{k-2}-\cdots-p_2q_2^{k-3}F_1}{f_{k-1}}\right] - q_2\left[\frac{F_k-p_2F_{k-1}-\cdots-p_2q_2^{k-2}F_1}{f_k}\right] - \frac{c_1}{c_2}.$$

Then $$\frac{f_k}{f_{k-1}} = \frac{f_k\left[\delta_{k-1}+\dfrac{c_1}{c_2}\right] + q_2\,[F_k-p_2F_{k-1}-\cdots-p_2q_2^{k-2}F_1]}{F_{k-1}-p_2F_{k-2}-\cdots-p_2q_2^{k-3}F_1}.$$

Thus $f_k/f_{k-1} > q_2^2$ if and only if

$$f_k \left[\delta_{k-1}+\frac{c_1}{c_2}\right] + q_2\ [F_k-p_2F_{k-1}-\cdots-p_2q_2^{k-2}F_1] > q_2^2\ [F_{k-1}-p_2F_{k-2}-\cdots-p_2q_2^{k-3}F_1]$$

if and only if $f_k \left[\delta_{k-1}+\dfrac{c_1}{c_2}\right] > (q_2^2+ p_2q_2)F_{k-1}- q_2F_k = q_2(F_{k-1}-F_k)$.

The last inequality holds since the LHS is positive and the RHS is negative. ‖

Using (5.15) and (5.16) we can write γ_k as a function of x_k and δ_{k-1} as follows:

$$\gamma_k= \frac{p_2\dfrac{[F(x_k)-F(x_k-\delta_{k-1})]}{f(x_k)}+q_2\dfrac{f(x_k-\delta_{k-1})}{f(x_k)}\left[p_2\delta_{k-1}+\dfrac{c_1}{c_2}\right]-\dfrac{c_1}{c_2}}{1-q_2^2\dfrac{f(x_k-\delta_{k-1})}{f(x_k)}} \tag{5.17}$$

$$\equiv u(x_k,\delta_{k-1})\ ,\quad k \geq 1.$$

<u>Lemma 5.2</u>. At the optimum, if $x_k < m$ (the mode of f), then $\gamma_k < \gamma_{k-1}$, for $k > 1$.

<u>Proof</u>. Using the notation $F_k \equiv F(x_k)$ and $f_k \equiv f(x_k)$, from (5.17) we have

$$\gamma_k = \frac{p_2[F_k-F_{k-1}]+ q_2f_{k-1}\left[p_2\delta_{k-1}+ \dfrac{c_1}{c_2}\right]- \dfrac{c_1}{c_2}f_k}{f_k- q_2^2f_{k-1}}.$$

For $x_k < m$, clearly $F_k - F_{k-1} < f_k\delta_{k-1}$, thus

$$\gamma_k < \frac{p_2 f_k \delta_{k-1} + q_2 f_{k-1}\left[p_2 \delta_{k-1} + \dfrac{c_1}{c_2}\right] - \dfrac{c_1}{c_2} f_k}{f_k - q_2^2 f_{k-1}}.$$

Applying (5.16) to $p_2\delta_{k-1}$ on the RHS we get

$$\gamma_k < \frac{(\gamma_{k-1} - q_2\gamma_k)(f_k + q_2 f_{k-1}) - \dfrac{c_1}{c_2}(f_k - q_2 f_{k-1})}{f_k - q_2^2 f_{k-1}}.$$

Since $f_k > f_{k-1}$, the last term in the numerator is positive so that $\gamma_k < (\gamma_{k-1} - q_2\gamma_k)(f_k + q_2 f_{k-1})/(f_k - q_2^2 f_{k-1})$. Rearranging we get $\gamma_k < \gamma_{k-1}\left[\dfrac{1}{1+q_2}\right]\left[1 + q_2\dfrac{f_{k-1}}{f_k}\right]$. The desired result follows from our assumption $f_{k-1} < f_k$. ‖

The following two lemmas show how γ_k increases geometrically.

<u>Lemma 5.3</u>. At optimum, $\delta_k > \delta_{k-1}$ implies $\gamma_{k+1} > \gamma_k$, $k \geq 1$.

<u>Proof</u>. $\gamma_{k+1} - \gamma_k = u(x_{k+1}, \delta_k) - u(x_k, \delta_{k-1})$. The result follows when we show $\partial u/\partial x_k \geq 0$ and $\partial u/\partial \delta_{k-1} > 0$. For k=1, $\delta_0 = x_1$ and $\gamma_1 = p_2 F_1/f_1 - c_1/c_2$. From Barlow, Proschan, and Hunter (1963) p.1087, $F(x)/f(x)$ is increasing in x for a PF_2 density; thus $\partial u/\partial x_1 = \partial u/\partial \delta_0 > 0$. Next we prove the results for $k > 1$.

From Barlow and Proschan (1965) p. 229, for fixed δ_{k-1}, $[F(x_k)-F(x_k-\delta_{k-1})]/f(x_k)$ and $f(x_k-\delta_{k-1})/f(x_k)$ are nondecreasing

in x_k for a PF_2 density f. Thus the denominator of u is nonincreasing in x_k and the numerator of u is nondecreasing, hence $\partial u/\partial x_k \geq 0$.

To show $\partial u/\partial \delta_{k-1} > 0$ we must consider whether x_{k-1} is greater than the mode of f or not. In the first case assume $f'(x_{k-1}) \leq 0$. Taking the partial derivative,

$$\frac{\partial u}{\partial \delta_{k-1}} = \frac{[p_2(1+q_2)f_{k-1} - q_2 f'_{k-1}\left[p_2\delta_{k-1} + \frac{c_1}{c_2}\right]][f_k - q_2^2 f_{k-1}]}{[f_k - q_2^2 f_{k-1}]^2}$$
$$- \frac{[p_2(F_k - F_{k-1}) + q_2 f_{k-1}\left[p_2\delta_{k-1} + \frac{c_1}{c_2}\right] - \frac{c_1}{c_2} f_k][q_2^2 f'_{k-1}]}{[f_k - q_2^2 f_{k-1}]^2}.$$

Simplifying we get

$$\frac{\partial u}{\partial \delta_{k-1}} = \frac{p_2(1+q_2)f_{k-1}}{f_k - q_2^2 f_{k-1}} - q_2 f'_{k-1} \frac{p_2\delta_{k-1} + \frac{c_1}{c_2} + q_2 u(x_k, \delta_{k-1})}{f_k - q_2^2 f_{k-1}}. \tag{5.18}$$

The denominator is positive by Lemma 5.1, and by assumption $f'_{k-1} \leq 0$, thus

$$\frac{\partial u}{\partial \delta_{k-1}} \geq \frac{p_2(1+q_2)f_{k-1}}{f_k - q_2^2 f_{k-1}} > 0. \tag{5.19}$$

For the second case assume $f'_{k-1} > 0$. From (5.18) $\partial u/\partial \delta_{k-1} \geq 0$ if and only if

$$p_2(1+q_2)f_{k-1} \geq q_2 f'_{k-1}[p_2\delta_{k-1} + \frac{c_1}{c_2} + q_2 u(x_k, \delta_{k-1})].$$

Using (5.16), the above inequality is equivalent to

$$p_2(1+q_2)\frac{f_{k-1}}{f'_{k-1}} \geq q_2(\gamma_{k-1} + \frac{c_1}{c_2}).$$

By Lemma 5.2, $\gamma_{k-1} \leq \gamma_1$, $k > 1$. Since f is PF_2 and f' exists everywhere, we have $f_{k-1}/f_{k-1}' \geq f_1/f_1' > F_1/f_1$. Then applying (5.15) we get

$$q_2\left(\gamma_{k-1}+\frac{c_1}{c_2}\right) \leq q_2\left(\gamma_1+\frac{c_1}{c_2}\right) = q_2 p_2 \frac{F_1}{f_1} < p_2 q_2 \frac{f_{k-1}}{f_{k-1}'} < p_2(1+q_2)\frac{f_{k-1}}{f_{k-1}'}.$$

Therefore $\partial u/\partial \delta_{k-1} > 0$. ‖

Let y_0 be such that $f''(y) > 0$ for $y > y_0 > m$, (the mode of f). Such y_0 exists since $f(y) > 0$ for $y > 0$.

<u>Lemma 5.4</u>. At the optimum, for $a > 1$ and $x_{k-1} > y_0$, $\delta_k = a\delta_{k-1}$ implies $\gamma_{k+1} > a\gamma_k$, $k \geq 1$.

<u>Proof</u>. From Lemma 5.3, $\partial u/\partial \delta_{k-1} > 0$ and $\partial u/\partial x_k \geq 0$. We will prove that $\partial^2 u/\partial \delta_{k-1}^2 > 0$ then the desired conclusion obtains from

$$\gamma_{k+1} = u(x_{k+1}, \delta_k) = u(x_{k+1}, a\delta_{k-1}) \geq u(x_k, a\delta_{k-1}) > au(x_k, \delta_{k-1}) = a\gamma_k.$$

To prove $\partial^2 u/\partial \delta_{k-1}^2 > 0$, write

$$u(x_k, \delta_{k-1}) = v(x_k, \delta_{k-1}) + w(x_k, \delta_{k-1})u(x_k, \delta_{k-1}),$$

where

$$v(x_k, \delta_{k-1}) = p_2\frac{F(x_k) - F(x_k-\delta_{k-1})}{f(x_k)} + q_2\frac{f(x_k-\delta_{k-1})}{f(x_k)}\left(p_2\delta_{k-1}+\frac{c_1}{c_2}\right) + \frac{c_1}{c_2}$$

$$\text{and } w(x_k, \delta_{k-1}) = q_2^2\frac{f(x_k-\delta_{k-1})}{f(x_k)}.$$

Then taking partial derivatives,

$$\frac{\partial^2 u}{\partial \delta_{k-1}^2} = \frac{\partial^2 v}{\partial \delta_{k-1}^2} + \frac{\partial^2 w}{\partial \delta_{k-1}^2} u(x_k, \delta_{k-1}) + 2\frac{\partial w}{\partial \delta_{k-1}} \frac{\partial u}{\partial \delta_{k-1}} + \frac{\partial^2 u}{\partial \delta_{k-1}^2} w(x_k, \delta_{k-1}) .$$

Since $1 - w(x_k, \delta_{k-1})$ is positive by Lemma 5.1, $u(x_k, \delta_{k-1})$ is positive by assumption, and $\partial u / \partial \delta_{k-1}$ is positive by the previous Lemma, it is sufficient to show that $\partial^2 v / \partial \delta_{k-1}^2 > 0$, $\partial^2 w / \partial \delta_{k-1}^2 > 0$ and $\partial w / \partial \delta_{k-1} > 0$. Now by assumption

$$\frac{\partial^2 v}{\partial \delta_{k-1}^2} = q_2 \frac{f''(x_k - \delta_{k-1})}{f(x_k)} \left[p_2 \delta_{k-1} + \frac{c_1}{c_2} \right] - q_2 (3 - 2p_2) \frac{f'(x_k - \delta_{k-1})}{f(x_k)} > 0.$$

Likewise, as required,

$$\frac{\partial w}{\partial \delta_{k-1}} = -q_2^2 \frac{f'(x_k - \delta_{k-1})}{f(x_k)} > 0$$

$$\frac{\partial^2 w}{\partial \delta_{k-1}} = -q_2^2 \frac{f''(x_k - \delta_{k-1})}{f(x_k)} > 0$$

thus completing the proof. ‖

Next we present the theorem which proves that if the failure density is PF_2 then optimum inspection intervals are nonincreasing.

<u>Theorem 5.2</u>. Let f be a PF_2 failure density with f' existing everywhere. At the optimum, $\delta_{i-1} \geq \delta_i$ for $i \geq 1$.

<u>Proof</u>. If $\delta_{i-1} \geq a\delta_i$ for $i \geq 1$, then $\gamma_{i-1} \geq a\gamma_i$ for $i \geq 2$, since from (5.11) and (5.16) we have

$$\gamma_{i-1} - a\gamma_i = \sum_{j=0}^{\infty} p_2 q_2^j (\delta_{j+i-1} - a\delta_{j+i}) . \tag{5.20}$$

Consequently, if $\gamma_i > a\gamma_{i-1}$ for some i, then there is a

positive integer k such that $\delta_{i+k} > a\delta_{i+k-1}$.

Suppose for a contradiction that $\delta_i > \delta_{i-1}$. By Lemma 5.3, $\delta_i > \delta_{i-1}$ implies $\gamma_{i+1} > \gamma_i$. Hence from (5.20) for some j > i, $\delta_j > \delta_{j-1}$. Thus, for some j, $\delta_j = a\delta_{j-1}$ with a > 1 and $x_{j-1} > y_0$ where y_0 is such that $f''(y) > 0$ for $y > y_0 > m$. It then follows from Lemma 5.4 that $\gamma_{j+1} > a\gamma_j$. From (5.16)

$$\gamma_{j+1} - a\gamma_j = p_2(\delta_{j+1} - a\delta_j) + q_2(\gamma_{j+2} - a\gamma_{j+1}),$$

hence either $\gamma_{j+2} > a\gamma_{j+1}$ or $\delta_{j+1} > a\delta_j$; but, from Lemma 5.4, the latter also implies that $\gamma_{j+2} \geq a\gamma_{j+1}$. Consequently γ_i increases at least geometrically resulting in $\lim_{i\to\infty}\gamma_i = \infty$.

Next we show that $\lim_{i\to\infty}\gamma_i = \infty$ contradicts the fact that $\{x_k^*\}$ is an optimum policy. Note that x_{n+1}^* is the first optimum checking point for the conditional density $f(t+x_n^*)/\overline{F}(x_n^*)$. Let $\mu(x_n^*)$ denote the mean of this conditional density, which is finite by Barlow, Hunter, and Proschan (1963) p.1089. Hence the expected cost for the optimum first check using the conditional density is bounded above uniformly in n. But this cost is

$$\frac{c_1}{p_2} + c_2 \int_{x_n^*}^{x_{n+1}^*} \sum_{i=0}^{\infty} p_2 q_2^{i-1} (x_{n+i}^* - t) \frac{f(t)\,dt}{\overline{F}(x_n^*)}$$

$$= \frac{c_1}{p_2} + c_2 \int_{x_n^*}^{x_{n+1}^*} (z_{n+1}^* - t) \frac{f(t)\,dt}{\overline{F}(x_n^*)}$$

$$= \frac{c_1}{p_2} + c_2 \int_0^{\delta_n} (\gamma_n + z_n^* - x_n^* - t) \frac{f(t+x_n^*)\,dt}{\overline{F}(x_n^*)}$$

$$\to \frac{c_1}{p_2} + c_2[\gamma_n + (z_n^* - x_n^*) - \mu(x_n^*)] \to \infty \text{ since } \gamma_n \to \infty,$$

which gives the desired contradiction. ||

5.5.3. Computation of Optimal Intervals.

The next theorem allows us to obtain an algorithm to compute an optimum inspection sequence; it requires the following lemma.

Lemma 5.5. At the optimum, $d\gamma_{i-1}/dx_1 \geq 0$ and $dx_i/dx_1 > 0$ for $i > 1$. Moreover, if $x_{j-1} > m$, then

$$\frac{d\gamma_{i-1}}{dx_1} \geq \eta^{i-j} \frac{d\gamma_{j-1}}{dx_1} \quad for\ i \geq j$$

where $\eta = \min_{0 \leq n < k} \left\{ \frac{(1+q_2) f(x_{j+n-1})}{f(x_{j+n}) + q_2 f(x_{j+n-1})} \right\} > 1.$

Proof. We prove the first part of the lemma, $d\gamma_{i-1}/dx_1 \geq 0$ and $dx_i/dx_1 > 0$ for $i > 1$, by induction.

Since f is PF_2, the ratio $F(y)/f(y)$ is nondecreasing in y so that at the optimum, $d\gamma_1/dx_1 \geq 0$. For $i \geq 2$,

$$\frac{d\gamma_i}{dx_1} = \frac{\partial \gamma_i}{\partial x_i} \frac{dx_i}{dx_1} + \frac{\partial \gamma_i}{\partial \delta_{i-1}} \frac{d\delta_{i-1}}{dx_1}. \tag{5.21}$$

Moreover, $d\delta_{i-1}/dx_1 = dx_i/dx_1 - dx_{i-1}/dx_1$ and from (5.16),

$$\frac{d\gamma_i}{dx_1} = \frac{p_2}{q_2} \left[\frac{dx_{i-1}}{dx_1} - \frac{dx_i}{dx_1} + \frac{1}{p_2} \frac{d\gamma_{i-1}}{dx_1} \right].$$

Equating the above to (5.21) and solving for dx_i/dx_1 we get

$$\frac{dx_i}{dx_1} = \frac{\left[\dfrac{p_2}{q_2} + \dfrac{\partial\gamma_i}{\partial\delta_{i-1}}\right]\dfrac{dx_{i-1}}{dx_1} + \dfrac{1}{q_2}\dfrac{d\gamma_{i-1}}{dx_1}}{\dfrac{p_2}{q_2} + \dfrac{\partial\gamma_i}{\partial x_i} + \dfrac{\partial\gamma_i}{\partial\delta_{i-1}}}. \tag{5.22}$$

At the optimum $\gamma_i = u(x_i, \delta_{i-1})$; from Lemma 5.3 $\partial\gamma_2/\partial x_2 \geq 0$ and $\partial\gamma_2/\partial\delta_1 > 0$. Hence $dx_2/dx_1 > 0$.

Next we assume $dx_{i-1}/dx_1 > 0$ and $d\gamma_{i-2}/dx_1 \geq 0$. If $d\delta_{i-2}/dx_1 \geq 0$, then by (5.21) $d\gamma_{i-1}/dx_1 \geq 0$; if instead $d\delta_{i-2}/dx_1 < 0$, then by (5.16) $d\gamma_{i-1}/dx_1 \geq 0$ again. Also by (5.22) $dx_i/dx_1 > 0$. Hence, by induction, for $i > 1$, $d\gamma_{i-1}/dx_1 \geq 0$ and $dx_i/dx_1 > 0$.

To prove the second part of the lemma, let $x_{j-1} > m$ (mode of f) and $i \geq j$. First assume $d\delta_{j-1}/dx_1 \geq 0$. From Lemma 5.3 $\partial\gamma_j/\partial x_j \geq 0$, and from the first part of this lemma $dx_j/dx_1 > 0$. Applying this to (5.21) we have

$$\frac{d\gamma_j}{dx_1} = \frac{\partial\gamma_j}{\partial x_j}\frac{dx_j}{dx_1} + \frac{\partial\gamma_j}{\partial\delta_{j-1}}\frac{d\delta_{j-1}}{dx_1} \geq \frac{\partial\gamma_j}{\partial\delta_{j-1}}\frac{d\delta_{j-1}}{dx_1}.$$

From (5.19) in Lemma 5.3

$$\frac{\partial\gamma_j}{\partial\delta_{j-1}} = \frac{\partial u}{\partial\delta_{j-1}} \geq \frac{p_2(1+q_2)f(x_{j-1})}{f(x_j) - q_2^2 f(x_{j-1})}.$$

Then

$$\frac{d\gamma_j}{dx_1} \geq \frac{p_2(1+q_2)f(x_{j-1})}{f(x_j) - q_2^2 f(x_{j-1})}\frac{d\gamma_{j-1}}{dx_1}.$$

By (5.16)

$$\frac{d\gamma_j}{dx_1} \geq \frac{p_2(1+q_2)f(x_{j-1})}{f(x_j) - q_2^2 f(x_{j-1})}\left[\frac{1}{p_2}\frac{d\gamma_{j-1}}{dx_1} - \frac{q_2}{p_2}\frac{d\gamma_j}{dx_1}\right].$$

Rearranging, we get

$$\frac{d\gamma_j}{dx_1} \geq \left[\frac{(1+q_2)f(x_{j-1})}{f(x_j)+q_2 f(x_{j-1})}\right]\frac{d\gamma_{j-1}}{dx_1}.$$

Hence it follows that

$$\frac{d\gamma_{j+k}}{dx_1} \geq \prod_{n=0}^{k}\left[\frac{(1+q_2)f(x_{j+n-1})}{f(x_{j+n})+q_2 f(x_{j+n-1})}\right]\frac{d\gamma_{j-1}}{dx_1}.$$

Let $\eta = \min\limits_{0\leq n<k}\left\{\frac{(1+q_2)f(x_{j+n-1})}{f(x_{j+n})+q_2 f(x_{j+n-1})}\right\}$.

By assumption $x_{j-1} > m$ implies $f(x_{j+n-1})/f(x_{j+n}) > 1$ for $n\geq 0$, which in turn implies $\eta > 1$. Then

$$\frac{d\gamma_{j+k}}{dx_1} \geq \eta^{k+1}\frac{d\gamma_{j-1}}{dx_1}.$$

Now let $i-1 = j+k$ and we obtain the desired result assuming $d\delta_j/dx_1 \geq 0$.

Next assume $d\delta_j/dx_1 < 0$. Then from (5.16)

$$0 > \frac{d\delta_j}{dx_1} = \frac{1}{p_2}\frac{d\gamma_{j-1}}{dx_1} - \frac{q_2}{p_2}\frac{d\gamma_j}{dx_1}.$$

Rearranging we get $\frac{d\gamma_j}{dx_1} > \frac{1}{q_2}\frac{d\gamma_{j-1}}{dx_1}$.

Hence it follows that $\frac{d\gamma_{j+k}}{dx_1} > \left(\frac{1}{q_2}\right)^{k+1}\frac{d\gamma_{j-1}}{dx_1}$.

By Lemma 5.1, for $n \geq 0$, $\frac{1}{q_2} > \frac{(1+q_2)f(x_{j+n-1})}{f(x_{j+n}) + q_2 f(x_{j+n-1})}$, which

implies $1/q_2 > \eta$, thus $\frac{d\gamma_{j+k}}{dx_1} > \eta^{k+1}\frac{d\gamma_{j-1}}{dx_1}$.

Again let $i-1 = j+k$ to obtain the desired result. ‖

Theorem 5.3. Let x_1^* be the optimal first inspection, and $\{x_i\}$ be the schedule obtained by (5.2) from an arbitrary initial x_1. Then:

a. if $x_1 > x_1^*$ there is an i such that $\delta_i > \delta_{i-1}$;

b. if $x_1 < x_1^*$ there is an i such that $\delta_i < 0$.

Proof. Let m be the mode of f and j be such that $x_{j-1} > m$. From Lemma 5.5, $d\gamma_{i-1}/dx_1 \geq 0$, $i > 1$, and

$$\frac{d\gamma_{i-1}}{dx_1} \geq \eta^{i-j}\frac{d\gamma_{j-1}}{dx_1}, \quad \textit{where } \eta > 1,$$

for all $i \geq j$. Therefore, an increase in x_1 from x_1^* will result in an increase in γ_{j-1} and γ_{i-1}. The latter, by Lemma 5.5, can be made arbitrarily large by appropriately choosing i. Therefore, for some i, $\gamma_i > \gamma_1$, which, from (5.20) implies that $\delta_{i+k} > \delta_{1+k}$ for some k. Likewise a decrease in x_1 from x_1^* will eventually result in a negative γ_i entailing a negative δ_{i+k}. ‖

As a result of Theorem 5.3, the optimal solution is unique. An optimum checking sequence can be obtained from (5.2) recursively once the first inspection time x_1 is obtained, and the bisection algorithm to find the optimum x_1 suggested by Barlow, Hunter, and Proschan (1963) for $p_2=1$ can be extended to the case for $0 < p_2 < 1$. If the choice of x_1 is too small, then eventually the checking intervals generated by (5.2) will be negative. If x_1 is too large, then checking intervals will eventually increase, and thus are not optimal.

Alternatively an optimum sequence can be obtained from (5.12) and (5.15). Suppose a starting x_1 is fixed. Then using (5.12) and (5.15) we obtain z_2 from

$$z_2 = x_1 + \frac{F(x_1)}{f(x_1)} - \frac{c_1}{p_2 c_2}.$$

Using (5.12) again we obtain z_1 from z_2 and x_1. Then the following steps can be repeated:

1. When the values of z_i, z_{i-1}, and x_{i-1} are known, use (5.12) to express the right hand side of (5.15) as a function of x_{i-1}, z_i, and z_{i-1}, and then solve for z_{i+1}.

2. Compute x_i using (5.12).

Then to obtain the optimum inspection sequence $\{x_i^*\}$ we apply the following algorithm.

<u>Algorithm 5.1</u>.

1. Compute $\{x_k\}$ and $\{z_k\}$ as above.

2. If for some k, $\gamma_k > \gamma_{k-1}$, reduce x_1 and repeat; if $\gamma_k < 0$, increase x_1 and repeat.

3. Continue until $\{x_k\}$ is determined to degree of accuracy required.

In this way both $\{x_k\}$ and $\{z_k\}$ can be determined. also at each iteration, (5.13) can be used to compute the terms of Cp_2, the expected cost.

<u>Example 5.3</u>. Let f be a Weibull density with α=2, λ=.01 where $f(t) = \lambda\alpha(\lambda t)^{\alpha-1}\exp[-(\lambda t)^{\alpha}]$.

Choose an inspection cost of c_1=10 and failure cost of c_2=1. Table 11 shows optimal first checking times, x_1^*, and expected costs, Cp_2, for various failure-detection probabilities, p_2.

Table 11. Optimal First Checking Times and Costs

p_2	Cp_2	x_1^*
1.0	42.227	68.15750
.9	46.237	68.87350
.8	50.789	70.02767
.7	56.104	71.69260

Table 12 below gives the optimal checking schedules $\{x_i^*\}$ and inspection intervals $\{\delta_i^*\}$ for p_2=1.0, .9, .8, and .7. As p_2 decreases, the first checking time is delayed

Table 12. Optimal Checking Schedules for a Weibull Density with λ=.01, α=2, c_1=10, c_2=1, p_2=1.0, .9, .8, and .7

	p_2=1.0		p_2=.9	
i	x_i^*	δ_{i-1}^*	x_i^*	δ_{i-1}^*
1	68.157	68.157	68.874	68.874
2	101.534	33.377	99.093	30.219
3	129.052	27.518	124.013	24.920
4	153.384	24.332	146.029	22.016
5	175.597	22.213	166.106	20.077
6	196.254	20.657	184.757	18.651
7	215.698	19.445	202.295	17.538
8	234.160	18.462	218.929	16.634
9	251.801	17.641	234.809	15.879
10	268.742	16.942	250.044	15.235
11	285.077	16.334	264.721	14.677
12	300.877	15.800	278.907	14.186

	p_2=.8		p_2=.7	
i	x_i^*	δ_{i-1}^*	x_i^*	δ_{i-1}^*
1	70.028	70.028	71.693	71.693
2	97.240	27.212	95.970	24.277
3	119.766	22.526	116.207	20.238
4	139.684	19.918	134.144	17.937
5	157.849	18.165	150.519	16.375
6	174.720	16.871	165.734	15.215
7	190.580	15.860	180.038	14.304
8	205.617	15.037	193.601	13.562
9	219.968	14.351	206.541	12.940
10	233.733	13.766	218.949	12.408
11	246.994	13.261	230.894	11.945
12	259.816	12.822	242.432	11.538

longer, but subsequent intervals are shorter.

Example 5.4. For a skewed left Weibull density, as p_2 increases the first checking time is delayed longer as seen in this example. Let f be a Weibull density with $\alpha=5$, $\lambda=.20$, $c_1=1$, and $c_2=250$. The optimal x_1 is 1.106 at $p_2=.25$ and 1.388 at $p_2=.50$.

5.5.4. Majorization Results for Finite Inspection Sequences.

Next we will show that for a fixed $k < \infty$ inspection sequences $\{x_k\}$ generated by (5.2) are related by weak majorization. To simplify notation we will use $F_k \equiv F(x_k)$ and $f_k \equiv f(x_k)$. Also from (5.2) an equivalent form for N_k is

$$N_k \equiv \sum_{i=1}^{k} \frac{(F_i - F_{i-1})\, q_2^{k-i}}{f_k} = \frac{F_k - p_2 \sum_{i=1}^{k-1} F_{k-i} q_2^{i-1}}{f_k}.$$

Also we will show the long-run expected cost function, C_{p_2}, is an order preserving function. An interesting consequence is that a minimum cost finite checking sequence is not obtained by truncating an optimum infinite sequence. Some definitions and theorems on majorization from Marshall and Olkin (1979) are repeated here for convenience. For $\underline{x} \in \mathbb{R}^n$, $x_{(i)}$ denotes the increasing order of components of $\underline{x}$,

i.e. $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$ and $x_{[i]}$ denotes the decreasing order, i.e., $x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$.

<u>Definition 5.1</u>. For $\underline{x},\underline{y} \in \mathbb{R}^n$,

$\underline{x}$ is weakly submajorized by $\underline{y}$, denoted

$$\underline{x} \prec_w \underline{y} \text{ if } \sum_1^k x_{[i]} \le \sum_1^k y_{[i]}, \text{ for } k=1,\ldots,n, \text{and } \underline{x} \text{ is weakly}$$

supermajorized by $\underline{y}$, denoted

$$\underline{x} \prec^w \underline{y} \text{ if } \sum_1^k x_{(i)} \ge \sum_1^k y_{(i)}, \text{ for } k=1,\ldots,n.$$

In either case $\underline{x}$ is said to be weakly majorized by $\underline{y}$ or $\underline{y}$ weakly majorizes $\underline{x}$. One simple consequence of the definition, from p.11 of Marshall and Olkin (1979), is that

$$x_i \le y_i \text{ for } i=1,\ldots,n \Rightarrow \underline{x} \prec_w \underline{y} \text{ and } \underline{y} \prec^w \underline{x}.$$

We will show that this is the case for inspection sequences generated by (5.2). The proof is by induction, and requires the following lemma.

<u>Lemma 5.6</u>. Let f be a PF_2 density and f' exist everywhere; then for k=1,2,...

$$\frac{\partial N_k}{\partial x_k} = 1 - \frac{f_k'}{f_k} N_k = 1 - \frac{f_k'}{f_k^2}\left[F_k - p_2 \sum_{i=1}^{k-1} F_{k-i} q_2^{i-1}\right] > 0.$$

Proof. Case 1: If $x_k <$ mode of f, then $f_k' > 0$ and $f_k/f_k' \geq F_k/f_k$. Then

$$\frac{f_k'}{f_k}\left[\frac{F_k}{f_k} - p_2\sum_{i=1}^{k-1}\frac{F_{k-i}}{f_k}q_2^{i-1}\right] \leq \frac{f_k'}{f_k}\left[\frac{f_k}{f_k'} - p_2\sum_{i=1}^{k-1}\frac{F_{k-i}}{f_k}q_2^{i-1}\right]$$

$$= 1 - p_2\sum_{i=1}^{k-1}\frac{f_k' F_{k-i}}{f_k^2}q_2^{i-1} < 1.$$

Case 2: If $x_k \geq$ mode of f, then $f_k' \leq 0$ and it suffices to show that $F_k > p_2\sum_{i=1}^{k-1}F_{k-i}q_2^{i-1}$. Clearly

$$p_2\sum_{i=1}^{k-1}F_{k-i}q_2^{i-1} < p_2F_k-1\sum_{i=1}^{k-1}q_2^{i-1} = p_2F_{k-1}\frac{(1-q_2^{k-1})}{(1-q_2)} = F_{k-1}(1-q_2^{k-1}) < F_{k-1}.$$

Therefore $F_k > F_{k-1} > p_2\sum_{i=1}^{k-1}F_{k-i}q_2^{i-1}$. ∥

Theorem 5.4. Let $\{x_n\}$ and $\{y_n\}$ be two inspection sequences generated by (5.2). Let $\underline{x}$ and $\underline{y}$ in $\mathbb{R}^{k+1}$ be the first k+1 inspection times for these sequences. When f is a PF_2 density and $p_2=1$, or when f is an exponential density and $0 \leq p_2 \leq 1$, then $x_1 \leq y_1$ implies $\underline{x} \prec_w \underline{y}$.

Proof. The proof is by induction. For n=1,

$$\delta_1 \equiv x_2 - x_1 = \frac{F_1}{f_1} - q_2\left[\frac{F_2 - p_2F_1}{f_2}\right] - \frac{c_1}{c_2}.$$

By implicit differentiation we find

$$\frac{dx_2}{dx_1} = \frac{1 + \frac{\partial N_1}{\partial x_1} + p_2 q_2 \frac{f_1}{f_2}}{1 + q_2 \frac{\partial N_2}{\partial x_2}},$$

which is positive by Lemma 5.6. Thus x_2 increases when x_1 increases.

Furthermore $dx_2/dx_1 \geq 1$ when $p_2 q_2 \frac{f_1}{f_2} + \frac{\partial N_1}{\partial x_1} \geq q_2 \frac{\partial N_2}{\partial x_2}$. This is true by Lemma 5.6 when $p_2=1$ ($q_2=0$). Combining this condition with the definition of δ_1 we get the equivalent condition

$$p_2 q_2 \frac{f_1}{f_2} + p_2 \geq \frac{f_2'}{f_2}\left[\delta_1 + \frac{c_1}{c_2}\right] + \frac{F_1}{f_1}\left[\frac{f_1'}{f_1} - \frac{f_2'}{f_2}\right]$$

which is clearly satisfied for an exponential density since LHS > 0 and RHS < 0. Now since $\delta_1 = x_2 - x_1$, we have

$$\frac{d\delta_1}{dx_1} = \frac{dx_2}{dx_1} - 1 \geq 0.$$

Next we will show $d\delta_2/dx_1 \geq 0$ and $dx_3/dx_1 \geq 1$. By definition

$$\delta_2 \equiv x_3 - x_2 = \frac{F_2 - p_2 F_1}{f_2} - q_2\left[\frac{F_3 - p_2 F_2 - P_2 q_2 F_1}{f_3}\right] - \frac{c_1}{c_2}$$

$$= \frac{F(x_2) - p_2 F(x_2 - \delta_1)}{f(x_2)} - q_2\left[\frac{F(x_2+\delta_2) - p_2 F(x_2) - p_2 q_2 F(x_2 - \delta_1)}{f(x_2+\delta_2)}\right] - \frac{c_1}{c_2}.$$

Then we can find $d\delta_2/dx_1$ by $\frac{d\delta_2}{dx_1} = \frac{\partial \delta_2}{\partial \delta_1}\frac{d\delta_1}{dx_1} + \frac{\partial \delta_2}{\partial x_2}\frac{dx_2}{dx_1}$.

First we find $\dfrac{\partial \delta_2}{\partial \delta_1} = \dfrac{p_2\left(\dfrac{f_1}{f_2} - q_2^2\dfrac{f_1}{f_3}\right)}{1 + q_2\dfrac{\partial N_3}{\partial x_3}} > 0$

since the numerator is positive by Lemma 5.1 and the denominator is positive by Lemma 5.6.

Next we find

$$\frac{\partial \delta_2}{\partial x_2} = \frac{\dfrac{\partial N_2}{\partial x_2} - q_2\dfrac{\partial N_3}{\partial x_3} + p_2 q_2\dfrac{f_2}{f_3} + p_2 q_2^2\dfrac{f_1}{f_3}}{1 + q_2\dfrac{\partial N_3}{\partial x_3}}.$$

The denominator is positive by Lemma 5.6. The numerator is positive when

$$p_2 q_2\frac{f_2}{f_3} + p_2 q_2^2\frac{f_1}{f_3} + \frac{\partial N_2}{\partial x_2} \geq q_2\frac{\partial N_3}{\partial x_3},$$

which is true by Lemma 5.6 when $p_2=1$ ($q_2=0$).

An equivalent condition is

$$p_2 q_2\frac{f_2}{f_3} + p_2 q_2^2\frac{f_1}{f_3} + p_2 \geq \frac{f_3'}{f_3}\left(\delta_2 + \frac{c_1}{c_2}\right) + \left(\frac{F_2 - p_2 F_1}{f_2}\right)\left(\frac{f_2'}{f_2} - \frac{f_3'}{f_3}\right),$$

which is clearly satisfied for an exponential density.

Therefore $d\delta_2/dx_1 \geq 0$ and $x_3 = x_1 + \delta_1 + \delta_2$ gives us $\dfrac{dx_3}{dx_1} = 1 + \dfrac{d\delta_1}{dx_1} + \dfrac{d\delta_2}{dx_1} \geq 1$. So an increase in x_1 results in increases in both x_2 and x_3.

Now assume $d\delta_{n-1}/dx_1 \geq 0$ and $dx_n/dx_1 \geq 1$ for $n=2,\ldots,k$.

To show $d\delta_k/dx_1 \geq 0$ and $dx_{k+1}/dx_1 \geq 1$. By definition

$$\delta_k = \frac{F(x_k) - p_2F(x_k-\delta_{k-1}) - p_2q_2F(x_k-\delta_{k-1}-\delta_{k-2}) - \cdots - p_2q_2^{k-2}F(x_k-\delta_{k-1}-\cdots-\delta_1)}{f(x_k)}$$

$$- q_2\frac{F(x_k+\delta_k) - p_2F(x_k) - p_2q_2F(x_k-\delta_{k-1}) - \cdots - p_2q_2^{k-1}F(x_k-\delta_{k-1}-\cdots-\delta_1)}{f(x_k+\delta_k)} - \frac{c_1}{c_2}.$$

Then we can find $d\delta_k/dx_1$ from

$$\frac{d\delta_k}{dx_1} = \left(\frac{\partial\delta_k}{\partial\delta_{k-1}}\right)\frac{d\delta_{k-1}}{dx_1} + \left(\frac{\partial\delta_k}{\partial\delta_{k-2}}\right)\frac{d\delta_{k-2}}{dx_1} + \cdots + \left(\frac{\partial\delta_k}{\partial\delta_1}\right)\frac{d\delta_1}{dx_1} + \left(\frac{\partial\delta_k}{\partial x_k}\right)\frac{dx_k}{dx_1}.$$

First we find for j=1,2,...,(k-1)

$$\frac{\partial\delta_k}{\partial\delta_{k-j}} = \frac{p_2\left(\frac{1}{f_k} - q_2^2\frac{1}{f_{k+1}}\right)\sum_{i=j}^{k-1} f_{k-i}q_2^{i-1}}{1 + q_2\frac{\partial N_{k+1}}{\partial x_{k+1}}} > 0$$

since the numerator is positive by Lemma 5.1 and the denominator is positive by Lemma 5.6. Next we find

$$\frac{\partial\delta_k}{\partial x_k} = \frac{\frac{\partial N_k}{\partial x_k} - q_2\frac{\partial N_{k+1}}{\partial x_{k+1}} + p_2q_2\sum_{i=1}^{k}\frac{f_{k+1-i}}{f_{k+1}}q_2^{i-1}}{1 + q_2\frac{\partial N_{k+1}}{\partial x_{k+1}}}.$$

The denominator is positive by Lemma 5.6. The numerator is positive when

$$p_2q_2\sum_{i=1}^{k}\frac{f_{k+1-i}}{f_{k+1}}q_2^{i-1} + \frac{\partial N_k}{\partial x_k} \geq q_2\frac{\partial N_{k+1}}{\partial x_{k+1}},$$

which is true by Lemma 5.6 when $p_2=1$ $(q_2=0)$. An equivalent

condition is

$$p_2 q_2 \sum_{i=1}^{k} \frac{f_{k+1-i}}{f_{k+1}} q_2^{i-1} + p_2 \geq \frac{f'_{k+1}}{f_{k+1}} \left[\delta_k + \frac{c_1}{c_2}\right] + N_k \left[\frac{f'_k}{f_k} - \frac{f'_{k+1}}{f_{k+1}}\right],$$

which is clearly satisfied by an exponential density. Thus $\partial\delta_k/\partial x_k \geq 0$, and therefore $d\delta_k/dx_1 \geq 0$. Also $x_{k+1} = x_1 + \delta_1 + \delta_2 + \cdots + \delta_k$ implies

$$\frac{dx_{k+1}}{dx_1} = 1 + \frac{d\delta_1}{dx_1} + \frac{d\delta_2}{dx_1} + \cdots + \frac{d\delta_k}{dx_1} \geq 1.$$

Thus an increase in x_1 results in increases in $x_2, \ldots, x_{k+1}$. Therefore $x_1 \leq y_1$ implies $x_i \leq y_i$ for $i=1,2,\ldots,k+1$ and we have $\underline{x} \prec_w \underline{y}$. $\|$

To show that C_{p_2}, the long-run expected cost function preserves the ordering of weak majorization, we need the following theorem of Ostrowski from Marshall and Olkin (1979) p. 59.

Theorem (Ostrowski, 1952). Let ϕ be a real-valued function, defined and continuous on $\mathscr{E}$, and continuously differentiable on the interior of $\mathscr{E}$, where $\mathscr{E}=\{\underline{x}: x_1 \leq \ldots \leq x_n\}$. Then

$\phi(\underline{x}) \leq \phi(\underline{y})$ whenever $\underline{x} \prec_w \underline{y}$ on $\mathscr{E}$ if and only if

$$\frac{\partial\phi}{\partial x_n}(\underline{z}) \geq \frac{\partial\phi}{\partial x_{n-1}}(\underline{z}) \geq \cdots \geq \frac{\partial\phi}{\partial x_1}(\underline{z}) \geq 0,$$

i.e., the gradient $\nabla\phi(z) \in \mathscr{E}_+$, for all $\underline{z}$ in the interior of $\mathscr{E}$. Similarly,

$\phi(\underline{x}) \le \phi(\underline{y})$ whenever $\underline{x} \prec^w \underline{y}$ on $\mathscr{E}$ if and only if

$$0 \ge \frac{\partial\phi}{\partial x_n}(\underline{z}) \ge \frac{\partial\phi}{\partial x_{n-1}}(\underline{z}) \ge \cdots \ge \frac{\partial\phi}{\partial x_1}(\underline{z})$$

for all $\underline{z}$ in the interior of $\mathscr{E}$.

<u>Theorem 5.5</u>. Let $\underline{x}$ and $\underline{y}$ in $\mathbb{R}^n$ be inspection schedules generated by (5.2). If $x_1 < y_1$ then $C_{p_2}(\underline{x}) \le C_{p_2}(\underline{y})$, when the conditions of Theorem 5.4 are satisfied.

<u>Proof</u>. For any inspection sequence $\underline{x}$ generated by (5.2) we have $\partial C_{p_2}/\partial x_i = 0$ for i=1,2,...,n. By Theorem 5.4 we have $\underline{x} \prec_w \underline{y}$, and applying Ostrowski's Theorem, $C_{p_2}(\underline{x}) \le C_{p_2}(\underline{y})$. ‖

A practical application of Theorems 5.4 and 5.5 involves choosing a finite checking sequence with minimum cost, C_{p_2}. Since C_{p_2} is an order preserving function, if we desire a minimum cost checking sequence of n inspection times, then we should choose an initial inspection time x_1 that is less than x_1^*, the optimal initial inspection time for an infinite sequence of inspections. Then $\underline{x} \prec_w \underline{x}^*$ and $C_{p_2}(\underline{x}) \le C_{p_2}(\underline{x}^*)$.

Since $x_1 < x_1^*$, the checking sequence generated from x_1

will eventually result in a negative inspection time, and thus a finite number of positive inspection times is obtained. The least cost initial value of x_1 can be chosen so that a checking sequence of exactly n inspection times is obtained, or x_1 can be chosen so that the checking sequence terminates at a specified inspection time.

As a consequence of choosing $x_1 < x_1^*$, we get smaller inspection intervals than those obtained by using x_1^*. Thus in the case f is an exponential density, $0 < p_2 \leq 1$, if we are interested in a least cost finite checking sequence, we will find that our inspection intervals will be decreasing as opposed to constant inspection intervals obtained for an optimal *infinite sequence.*

Example 5.5. Let f be an exponential density with $\lambda=.01$, $f(t)=\lambda\exp(-\lambda t)$, $t\geq 0$. Choose a cost per inspection, $c_1=10$, cost per unit of failure time, $c_2=1$, and probability of detecting a failure, $p_2=.9$. For an optimal infinite checking sequence, $x_1^*=41.874$ and constant checking intervals are $\delta_k^*=36.805$ resulting in a cost of $C_{p_2} = 57.075$.

To obtain an optimal finite schedule containing n=10 inspection times, observe from Table 13 that an initial inspection time of $x_1=39$ results in a checking sequence of only 9 inspection times; whereas, an initial inspection time of $x_1=40$ results in a checking sequence of 10

inspection times. Also an initial inspection time of x_1=39.9 also results in a checking sequence of 10 inspection times.

The cost of the 10-inspection sequence beginning at x_1=40 is C_{p_2} = 18.898. The cost of the 10-inspection sequence beginning at x_1=39.9 is C_{p_2} = 18.669. Thus the optimal sequence for n=10 begins at x_1, where 39 < x_1 < 39.9.

Table 13. Inspection Schedules and Intervals for p_2=.9 for an Exponential Density λ=.01, c_1=10, c_2=1

	Cp_2=5.535		Cp_2=18.669		Cp_2=18.898	
i	x_i	δ_{i-1}	x_i	δ_{i-1}	x_i	δ_{i-1}
1	39.00	39.00	39.90	39.90	40.00	40.00
2	72.11	33.11	74.16	34.26	74.39	34.39
3	103.66	31.55	107.32	33.17	107.73	33.35
4	133.05	29.39	138.95	31.63	139.61	31.88
5	159.49	26.44	168.45	29.50	169.46	29.85
6	181.99	22.50	195.04	26.59	196.51	27.05
7	199.42	17.42	217.73	22.70	219.83	23.32
8	210.55	11.14	235.40	17.67	238.28	18.46
9	214.32	3.77	246.83	11.43	250.67	12.39
10	210.02	-4.30	250.93	4.11	255.87	5.20
11			246.96	-3.98	253.10	-2.78

6. IMPERFECT DETECTION OF SOFTWARE FAULTS

6.1. Introduction.

In this chapter we present the impact of Type II inspection error on two software fault detection models, where the probability of detecting a fault is $0 < p_2 \leq 1$. In Section 6.2 we introduce Type II error into the Koch and Kubat (1983) optimal release time model. In Section 6.3 we explore fault diversity in the Jelinski-Moranda (JM) model.

6.2. Imperfect Fault-Detection Model.

The Koch and Kubat (1983) software release time decision model is based on the following function for the average number of faults detected during testing:

$$E[N(T)] = M[1-\exp(-\theta T)] \tag{6.1}$$

where T = software release time (total testing time)

M = total number of faults in the software prior to testing

θ = proportionality constant.

Next the optimal time, T, to release software for operational use is obtained by maximizing the average gain

function, G(T):

$$G(T) = cM[1-\exp(-\theta T)] - c_1 T \quad (6.2)$$

where c_1 = cost per rate of testing

c_2 = cost of correcting a fault during test phase

c_3 = cost of correcting a fault during operational phase ($c_3 > c_2$)

$c = c_3 - c_2$.

The above cost factors presented by Bai and Yun (1988) are a condensed version of the many different cost factors used by Koch and Kubat (1983), and are used here for simplicity and ease of understanding the model. The optimal release time, T, obtained by maximizing the average gain is the same as the optimal release time obtained by minimizing the expected loss function used by Okumoto and Goel (1980).

Equation (6.1) is identical in form to the mean value function of the Goel-Okumoto (1979) nonhomogeneous Poisson process (NHPP) software reliability growth model (SRGM) with intensity function dE[N(T)]/dT. Yamada and Osaki (1985) categorize SRGM's according to whether their fault detection rate per fault is increasing, constant, or decreasing, where fault detection rate per fault is defined as $r(T) = \{dE[N(T)]\}/\{M-E[N(T)]\}$. For the Goel-Okumoto NHPP SRGM, $r(T) = \theta$, which is a constant fault detection rate per fault. Note that in both the Koch and Kubat model and the Goel-Okumoto NHPP SRGM that fault "detection" is

operationally equivalent to fault "occurrence" because it is assumed that every fault that occurs is immediately detected and corrected.

In the present treatment we assume that a fault that occurs may not be detected, with probability q_2. Then $p_2 = 1 - q_2$ is the probability that a fault which occurs is detected (and corrected immediately). This results in reducing the fault detection rate per fault, θ, so that the imperfect fault detection rate per fault is $p_2\theta$. Thus we modify (6.1), the average number of faults detected during a test period of duration T, to

$$E[N(T)] = M[1-\exp(-p_2\theta T)] \tag{6.3}$$

and the average gain (6.2) is modified as

$$G(T) = cM[1-\exp(-p_2\theta T)] - c_1 T. \tag{6.4}$$

We obtain the optimal release time $T^*_{p_2}$ by solving $dG(T)/dT = 0$ to get

$$T^*_{p_2} = \frac{1}{p_2\theta}\ln\left[\frac{p_2\theta cM}{c_1}\right]. \tag{6.5}$$

Since $d^2G(T)/dT^2 < 0$, $T^*_{p_2}$ maximizes the gain. To compare this optimal release time to the optimal release time T^*_1 obtained by Koch and Kubat when $p_2=1$, we rewrite (6.5) to get

$$T_{p_2}^* = \frac{1}{p_2\theta}\ln p_2 + \frac{1}{p_2}T_1^* . \tag{6.6}$$

From equations (6.5) and (6.6) we see that if $1/M\theta \geq c/c_1$, then the optimal release time is $T_{p_2}^* = 0$ for all p_2.

If $1/M\theta < c/c_1$, then the optimal release time is given by (6.6). For $1/M\theta < c/c_1$, Figure 6 compares $T_{p_2}^*$ to T_1^*. $T_{p_2}^*$ is a concave function of p_2 with maximum value of $T_{p_2}^* = cM/c_1e$ at $p_2 = c_1e/cM\theta$.

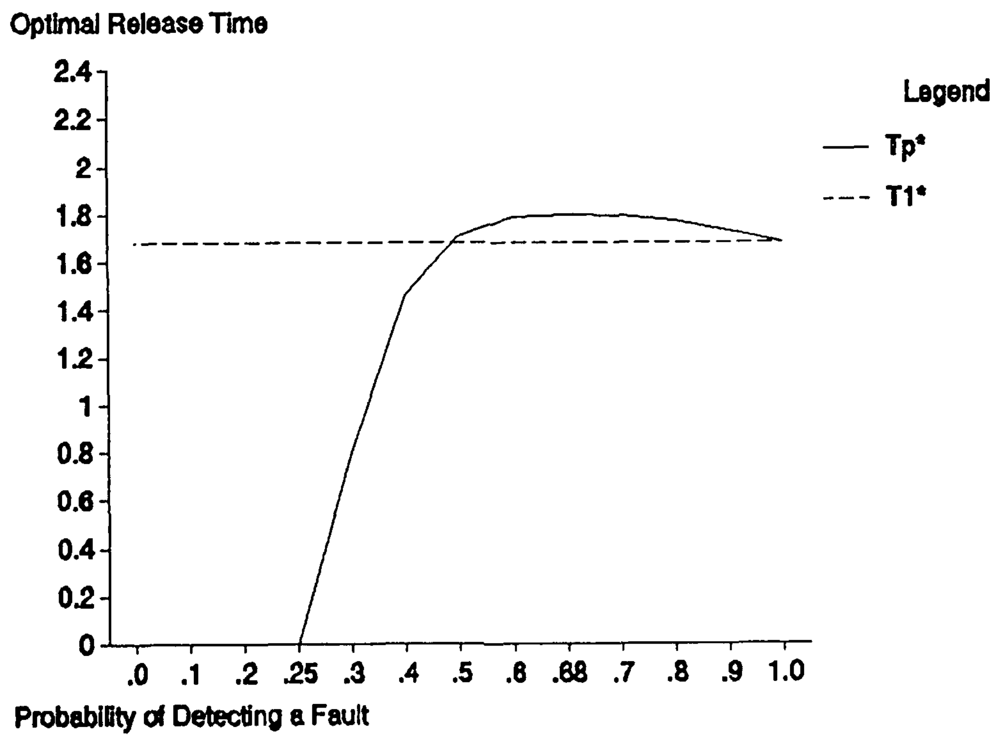

Figure 6. Optimal Release Time, T_p^*.

When $0<p_2\leq 1$, the optimum release time $T_{p_2}^*$ differs from the Koch and Kubat optimum T_1^* at all but two values of p_2, when $p_2=1$ and the value of p_2 that solves the following equation.

$$\ln p_2 = (p_2-1)\ln\left[\frac{cM\theta}{c_1}\right]. \tag{6.7}$$

From Figure 6 we see that when the probability of detecting a fault is smaller than the solution to (6.7), then the optimal release time $T_{p_2}^*$ is smaller than the perfect-detection optimal release time T_1^*; otherwise, $T_{p_2}^* \geq T_1^*$, when p_2 is greater than the solution to (6.7).

<u>Example 6.1</u>. A numerical example is provided to compare the results of this imperfect fault-detection model to the results obtained by Koch and Kubat using the values in their example 1: $c=3500$, $c_1=21500$, $M=30$, $\theta=5/6$. Then $T_1^*=1.68$, $cM/c_1e = 1.80$, $c_1e/cM\theta = .68$, $c_1/cM\theta = .25$. The graph of this particular example is shown in Figure 6.

<u>6.3. Fault Diversity in the Jelinski-Moranda Model</u>.

Many software models make the erroneous assumption that each fault is equally likely to cause failure (and thus equally likely to be detected). Boland, Proschan, and

Tong (1987) p. 185 present the impact of fault diversity in models of the Jelinski-Moranda (JM) type. They assume that N unknown faults in a software system have detection times, Z_i, which are independent exponential random variables with failure rates $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N)$. Assuming that when a fault occurs, it is detected and eliminated, the vector $\underline{\lambda}$ represents the diverse fault detection rates for the system. Define $T_{n,\underline{\lambda}}$ to be the time until any n faults in the system are eliminated. Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N)$ and $\underline{\lambda}' = (\lambda_1', \ldots, \lambda_N')$ be two vectors of fault detection rates for two software systems. Boland, Proschan, and Tong (1987) then prove that if $\underline{\lambda} >^m \underline{\lambda}'$ ($\underline{\lambda}$ majorizes $\underline{\lambda}'$) then $T_{n,\underline{\lambda}} \geq^{st} T_{n,\underline{\lambda}'}$. Thus the more diverse the fault detection rates are, the longer it takes to eliminate any n faults.

This section presents similar results specifically for the Jelinski-Moranda model and shows how fault diversity in the JM model is related to imperfect fault detection.

Assume there are N unknown faults initially in the system and times between fault detections are independent, exponentially distributed random variables Z_i. Then we know that in the JM model for Z_i, the time between the detection of the (i-1)st and the ith fault, the fault detection rate is $\lambda_i = (N-i+1)\lambda$, i=1,...,N, where λ is a proportionality constant. The model assumes that faults that occur are detected and corrected with certainty. We can characterize the JM model by the vector $\underline{\lambda}$ =

$(\lambda_1,\ldots,\lambda_N)$. For a given vector $\underline{\lambda}$, define the random variable $T_{n,\underline{\lambda}}$ to be the time until the first n faults detected in the system are eliminated.

To show the impact of fault diversity, we need the following definition and lemma from pages 251 and 256 of Ross (1983).

Definition 6.1. The random variable X is stochastically larger than the random variable Y, written $X \geq^{st} Y$ if $P[X>a] \geq P[Y>a]$ for all a.

Lemma 6.1. Let $X_1, X_2, \ldots, X_n$ be independent and $Y_1, Y_2, \ldots, Y_n$ be independent. If $X_i \geq^{st} Y_i$, then for any increasing f, $f(X_1,\ldots,X_n) \geq^{st} f(Y_1,\ldots,Y_n)$.

Theorem 6.1. Assuming the Jelinski-Moranda model, let $\underline{\lambda} = (\lambda_1,\ldots,\lambda_N)$ and $\underline{\lambda}' = (\lambda_1',\ldots,\lambda_N')$ be two vectors of fault detection rates for two software systems, where $\lambda_i = (N-i+1)\lambda$ and $\lambda_i' = (N-i+1)\lambda'$. If $\lambda \leq \lambda'$ then $T_{n,\underline{\lambda}} \geq^{st} T_{n,\underline{\lambda}'}$ for any $n \leq N$.

Proof. $\lambda_i = (N-i+1)\lambda \leq (N-i+1)\lambda' = \lambda_i'$ for all $i=1,\ldots,N$. Then $P[Z_i \geq a] = \exp(-\lambda_i a) \geq \exp(-\lambda_i' a) = P[Z_i' \geq a]$ so that $Z_i \geq^{st} Z_i'$. Define

$$f_n(Z_1,\ldots,Z_N) = \sum_{i=1}^{n} Z_i = T_{n,\underline{\lambda}}.$$

Since f_n is an increasing function, by Lemma 6.1 we have $T_{n,\underline{\lambda}} \geq^{st} T_{n,\underline{\lambda}'}$ for $n \leq N$. ||

Clearly from the definition of weak supermajorization in Chapter 5, if $\lambda \leq \lambda'$ then $\underline{\lambda} \succ^{w} \underline{\lambda}'$, thus for the JM model our result is similar to the Boland, Proschan, and Tong (1987) Theorem 5.2 p. 186 for models of the JM type which requires majorization instead of weak supermajorization to conclude $T_{n,\underline{\lambda}} \geq^{st} T_{n,\underline{\lambda}'}$.

From Theorem 6.1 we can conclude that if the fault detection rate λ_i for a software system is less than the fault detection rate λ_i' for another software system, then the time to correction of n faults for the first system, $T_{n,\underline{\lambda}}$ is longer than the $T_{n,\underline{\lambda}'}$ for the second system.

Now assume that the probability that a fault is detected upon occurrence is $0 < p_2 < 1$. Once the fault is detected it is immediately corrected. Suppose there are two teams working independently to correct faults in identical software systems. The first team detects faults with probability $0 < p_2 < 1$, while the second team detects faults with probability one. We can describe the fault correction rates for the two teams as $\lambda_i = p_2\lambda_i'$ and λ_i' respectively, then $T_{n,\underline{\lambda}} \geq^{st} T_{n,\underline{\lambda}'}$. Thus imperfect fault detection increases fault correction time, as expected.

REFERENCES

Abdel-Hameed, M. (1986), "Optimum Replacement of a System Subject to Shocks," Journal of Applied Probability, 23, 107-114.

Bai, D. S. and Yun, W. Y. (1988), "Optimum Number of Errors Corrected before Releasing a Software System," IEEE Transactions on Reliability, R-37, 41-44.

Barlow, R. E., Hunter, L. C. and Proschan, F. (1963), "Optimum Checking Procedures," Journal of the Society for Industrial and Applied Mathematics, 11, 1078-1095.

Barlow, R. E. and Proschan, F. (1965), Mathematical Theory of Reliability, New York: John Wiley and Sons, Inc.

Barlow, R. E. and Proschan, F. (1981), Statistical Theory of Reliability and Life Testing, Silver Spring, MD: McArdle Press, Inc.

Boland, P. J., Proschan, F. and Tong, Y. L. (1987), "Fault Diversity in Software Reliability," Probability in the Engineering & Informational Sciences, 1, 178-188.

Brown, M. and Proschan, F. (1983), "Imperfect Maintenance," IMS Lecture Notes, 2, 179-188.

Brown, M. and Proschan, F. (1983), "Imperfect Repair," Journal of Applied Probability, 20, 851-859.

Derman, C. (1961), "On Minimax Surveillance Schedules," Naval Research Logistics Quarterly, 8, 415-419.

Derman, C. and Sacks, J. (1960), "Replacement of Periodically Inspected Equipment," Naval Research Logistics Quarterly, 7, 597-607.

Dunn, R. and Ullman, R. (1982), Quality Assurance for Computer Software, New York: McGraw-Hill.

Fontenot, R. A. and Proschan, F. (1984), "Some Imperfect Maintenance Models," Reliability Theory and Models, eds. Abdel-Hameed, Çinlar, and Quinn, New York: Academic Press, Inc., pp. 83-101.

Goel, A. L. (1985), "Software Reliability Models: Assumptions, Limitations, and Applicability," IEEE Transactions on Software Engineering, SE-11, 1411-1423.

Goel, A. L. and Okumoto, K. (1978), "Bayesian Software Prediction Models," Technical Report 78-155, Rome Air Development Center, Griffiss AFB, New York.

Goel, A. L. and Okumoto, K. (1979), "Time-Dependent Error-Detection Rate Model for Software Reliability and Other Performance Measures," IEEE Transactions on Reliability, R-28, 206-211.

Herge, D. C., Proschan, F. and Sethuraman, J. (1986), "Optimal Replacement Age in an Imperfect Inspection Model," Technical Report M729, Florida State University, Dept. of Statistics.

Jelinski, Z. and Moranda P. (1972), "Software Reliability Research," Statistical Computer Performance Evaluation, ed. Freiberger, New York: Academic Press, pp. 465-484.

Kaio, N. and Osaki, S. (1984), "Some Remarks on Optimum Inspection Policies," IEEE Transactions on Reliability, R-33, 277-279.

Keller, J. B. (1974), "Optimum Checking Schedules for Systems Subject to Random Failure," Management Science, 21, 256-260.

Keller, J. B. (1982), "Optimum Inspection Policies," Management Science, 28, 447-450.

Klein, M. and Rosenberg, L. (1960), "The Deterioration of Inventory and Equipment," Naval Research Logistics Quarterly, 7, 49-62.

Koch, H. S. and Kubat, P. (1983), "Optimal Release Time of Computer Software," IEEE Transactions on Software Engineering, SE-9, 323-327.

Luss, H. and Kander, Z. (1974), "Inspection Policies when Duration of Checking is Non-Negligible," Operational Research Quarterly, 25, 299-309.

Marshall, A. W. and Olkin, I. (1979), Inequalities: Theory of Majorization and Its Applications, New York: Academic Press, Inc.

Marshall, A. W. and Proschan, F. (1972), "Classes of Distributions Applicable in Replacement, with Renewal Theory Implications," Proceedings of the Sixth Berkely

Symposium on Mathematical Statistics and Probability, I, 395-415.

Nakagawa, T. (1981), "Generalized Models for Determining Optimal Number of Minimal Repairs before Replacement," Journal of the Operations Research Society of Japan, 24, 325-337.

Okumoto, K. and Goel, A. L. (1980), "Optimum Release-Time for Software Systems Based on Reliability and Cost Criteria," Journal of System Software, 1, 315-318.

Park, K. S. (1988), "Optimal Continuous-Wear Limit Replacement under Periodic Inspections, IEEE Transactions on Reliability, R-37, 97-102.

Parmigiani, G. (1990a), "Inspecting for Failures while Learning about the Failure Rate," Carnegie-Mellon Univ. Preliminary Draft presented at 40th Seminar on Bayesian Inference in Econometrics and Statistics, Washington, D.C.

Parmigiani, G. (1990b), "On Optimal Screening Ages," Technical Report 497, Carnegie Mellon University, Dept. of Statistics.

Parmigiani, G. (1990c), "Optimal Inspection and Replacement Policies with Age-Dependent Failures and Fallible Tests," Technical Report 6, Carnegie Mellon University, Dept. of Statistics.

Parmigiani, G. (1990d), "Optimal Scheduling of Inspections, with an Application to Medical Screening Tests," Doctoral Thesis, Carnegie Mellon University, Dept. of Statistics.

Ross, S. M. (1983), Stochastic Processes, New York: John Wiley and Sons.

Savage, I. R. (1956), "Cycling," Naval Research Logistics Quarterly, 3, 163-175.

Sengupta, B. (1982), "An Exponential Riddle," Journal of Applied Probability, 19, 737-740.

Shooman M. and Natarajan S. (1977), "Effect of Manpower Development and Bug Generation on Software Error Models," Technical Report 76-400, Rome Air Development Center, Griffiss AFB, New York.

Taylor, H. M. (1975), "Optimal Replacement Under Additive Damage and Other Failure Models," Naval Research Logistics Quarterly, 22, 1-18.

Weiss, G. H. (1962), "A Problem in Equipment Maintenance," Management Science, 8, 266-277.

Yamada, S. and Osaki, S. (1985), "Software Reliability Growth Modeling: Models and Applications," IEEE Transactions on Software Engineering, SE-11, 1431-1437.

BIOGRAPHICAL SKETCH

Donna Carol (Kling) Herge was born on November 11, 1948 in Rockford, Illinois. In 1973 Donna Carol Kling married John Arthur Herge of Napoleon, Ohio. In 1989 their son Thomas William Arthur Herge was born.

In 1970 she graduated Phi Beta Kappa, Magna Cum Laude from Rockford College with a BA in Mathematics and Philosophy. She was awarded a Master of Science Degree in Mathematics from Wright State University in 1973 and a Master of Science degree in Statistics from Florida State University in 1985.

Commissioned as an officer in the United States Air Force in 1971, she has held various positions including Detachment Commander and Chief of Maintenance, first female instructor in the Department of Mathematical Sciences of the United States Air Force Academy from 1976 to 1980, and Statistics Branch Chief, Department of Mathematics and Statistics at the Air Force Institute of Technology from 1986 to 1991. Lieutenant Colonel Herge is currently Director of Research and Analysis for the Air Force Quality Center at Maxwell Air Force Base, Alabama.